國立成功大學79學年度工程科學研究所考試(工程數學 試題)

1. (a) Prove that $D^n(e^{\lambda x}g(x)) = [(D+\lambda)^n g(x)]\, e^{\lambda x}$ (8%)
   where $D \equiv d/dx$

   (b) Solve $(D-1)^n g(x) = e^x$ (5%)

2. (a) Find the particular solution of (8%)

   $\dot{y} + 3y = f(t)$

   where f(t) is a periodic function whose definition in one period is

   $$f(t) = \begin{cases} -1 & -1<t<0 \\ 1 & 0<t<1 \end{cases}$$

   (b) If the characteristic equation of the differential equation (8%)

   $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n)y = 0$

   has distinct roots λ_1, λ_2, ..., and λ_n where a_i, $i=1,2,\dots,n$ are constants, then what is the solution of

   $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n)y = f(t)$

   for the initial conditions $y=Dy=D^2y=\dots=D^{(n-1)}y=0$ at $t=0$

3. (a) Prove that $\cos(ix)=\cosh x$ (5%)

   (b) If f(z) is analytic throughout a closed connected region R, then at any interior point z_0 of R the derivatives of f(z) of all order exist and are analytic. Prove that (8%)

   $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$

   where C is the boundary of R.

4. Assume that (8%)

   (a) The solution of $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = 0$
   with $u(0,y)=u_0$, $u(x,0)=0$, $u(1,y)=0$ and $u(x,1)=0$ is $u_1(x,y)$.

   (b) The solution of $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = 0$
   with $u(x,0)=v_0$, $u(0,y)=0$, $u(1,y)=0$ and $u(x,1)=0$ is $u_2(x,y)$.

   (c) The solution of $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = Q(x,y)$
   with $u(0,y)=0$, $u(x,0)=0$, $u(1,y)=0$ and $u(x,1)=0$ is $u_3(x,y)$.
   where $x,y \in [0,1]$

   Question:
   What is the solution of $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = Q(x,y)$
   with $u(0,y)=0$, $u(x,0)=0$, $u(1,y)=u_0$ and $u(x,1)=v_0$?

5. (a) Prove that $\iint_R f(x,y)\,dxdy = \iint_{R'} f[x(u,v),y(u,v)]\; J\left(\frac{x,y}{u,v}\right) dudv$ (12%)

(b) and use $x=r\cos\theta$, $y=r\sin\theta$ to find the value of (8%)

$$\iint_R \cos(x^2+y^2)\,dxdy, \quad \text{where } R: x^2+y^2 \leqq 9, \quad x \geqq 0.$$

6. Matrix $A=\begin{pmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{pmatrix}$

(a) Find its eigenvalues and eigenvectors. (8%)
(b) Prove that these eigenvectors are independent and orthogonal. (8%)
(c) Transform A into a diagonal matrix B. (8%)

7. Solve $\iint_S \vec{F}\cdot\vec{n}\,d\sigma$, where $\vec{F}=(y^2+z^2)^{3/2}\vec{i} + \sin(x^2-z^2)\vec{j} + e^{x^2-y^2}\vec{k}$ (6%)

and S is the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $abc\neq 0$