1. If R is a plane region bounded by a simply closed curve, and if U(x,y), V(x,y), $\frac{\partial U}{\partial y}$ and $\frac{\partial V}{\partial x}$ are continuous at all points of R and its boundary C. Prove

$$\oint_C (Udx + Vdy) = \iint_R \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) dxdy$$

provided that the line integral is taken in the positive direction around C. (10 %)

2. Let U be a unitary matrix. Prove (15%)
   (a) U is normal
   (b) $\|U\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in C^n$
   (c) If λ is an eigen value of U, then $|\lambda| = 1$.

3. If C is a closed curve, and if f(z) is analytic except at a finite number of singular points z_1, ... and z_n in the interior of C. Prove (10%)

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_{r=1}^{n} \operatorname*{Res}_{z=z_r} f(z)$$

4. For the domain $(x, y) \in [0, 1] \times [0, 1]$, the following statements are known
   (a) the solution for $\nabla^2 u = 0$ with $u(0, y) = g_1(y)$, $u(1, y) = g_2(y)$, $u(x, 0) = 0$ and $u(x, 1) = 0$ is $u_1(x, y)$
   (b) the solution for $\nabla^2 u = 0$ with $u(0, y) = u(1, y) = 0$, $u(x, 0) = h_1(x)$ and $u(x, 1) = h_2(x)$ is $u_2(x,y)$
   (b) the solution for $\nabla^2 u = f(x, y)$ with $u(0, y) = 0$, $u(1, y) = 0$, $u(x, 0) = 0$ and $u(x, 1) = 0$ is $u_3(x, y)$
   Question:
   What is the solution for $\nabla^2 u = f(x, y)$ with $u(0, y) = h_1(y)$, $u(1,y) = h_2(y)$, $u(x, 0) = g_1(x)$ and $u(x, 1) = g_2(x)$? (15%)

5. Find the solution of the following differential equation, (10%)

$$y'' + \frac{1}{x}y' + 1 = 0,\quad y(1) = -\frac{1}{4} \text{ and } y'(1) = -\frac{1}{2}$$

6. Find the solution of the following equation, (10%)

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \qquad y(0,x) = 1,\ y(t,0) = 0,\ \lim_{x\to\infty} y = 1.$$

Hint: (a) Use $\eta = \frac{x}{2\sqrt{t}}$ to transform the partial differential equation into a differential equation.

(b) error function, $\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} \exp(-z^2)\,dz$; $\operatorname{erf}(\infty) = 1$.

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7. Solve the following initial value problem by using the method of Laplace transformation: (10%)

$$y'' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 3$$

8. Find the Fourier series of the periodic function f(x), (10%)

$$f(x) = \begin{cases} -1 \text{ when } -\pi < x < 0 \\ +1 \text{ when } 0 < x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$

9. Use the finite difference method to solve the following differential equation. Only three nodes are used in solving this problem. What are the values of y at x = 1.0 and 2.0? (10%)

$$y'' + y = 1, \quad y(0) = 2, \qquad y'(2) = -1$$