

1. Let  $A$  and  $B$  be  $3 \times 3$  matrices with  $\det(A) = 4$  and  $\det(B) = 5$ . Find the value of
- a)  $\det(2AB)$  (5%)
- b)  $\det(A^{-1}B)$  (5%)

2. Find the rank and the nullity of the matrix

$$A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \quad (10\%)$$

3. let  $b_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ,  $b_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $b_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ ,  $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and let  $L$  be the linear transformation from  $\mathbb{R}^2$  into  $\mathbb{R}^3$  defined by  $L(x) = x_1 b_1 + x_2 b_2 + (x_1 + x_2) b_3$

where  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Find the matrix  $A$  representing  $L$  with respect to the bases  $[e_1, e_2]$  and  $[b_1, b_2, b_3]$  (15%)

4. Let  $A$  be a Hermitian matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$  and orthonormal eigenvectors  $u_1, \dots, u_n$ .

Show that

$$A = \lambda_1 u_1 u_1^H + \lambda_2 u_2 u_2^H + \dots + \lambda_n u_n u_n^H. \quad (15\%)$$

(40)

5. Please answer the following questions briefly. (20%)

- (5.1) Is the shortest path joining two vertices necessarily unique?
- (5.2) Is the tree of shortest paths from a fixed vertex necessarily a minimum spanning tree?
- (5.3) Can Dijkstra's algorithm be applied to tell if an undirected graph is connected?
- (5.4) How many iterations does Warshall's algorithm perform on a directed graph with  $p$  vertices and  $q$  edges?
- (5.5) What is the significance of a one in the reachability matrix of a directed graph?
- (5.6) Simplify the Boolean function  
 $F(x,y,z) = xy'z + xy'z + x'yz + x'y'z'$
- (5.7) Assuming that  $a, b,$  and  $c$  are atoms, is  $((((ab)(bc))) * a) *$  (strictly speaking) a regular expression?
- (5.8) Assuming that  $a, b,$  and  $c$  are atoms, is  $((a+b) + (ac)) *$  (strictly speaking) a regular expression?
- (5.9) Is the graph whose adjacency matrix is given below planar?

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

- (5.10) If  $m$  pigeons are placed in  $n$  pigeonholes, and  $m < n$ , then some pigeonhole must contain more than one pigeon.
6. Suppose that  $A$  and  $B$  are disjoint graphs, both of which have an Euler cycle. Consider the graph  $G$  which is constructed by joining one of the vertices of  $A$  to one of the vertices of  $B$ . Does  $G$  have an Euler path or an Euler cycle? Explain. (5%)
7. Find a deterministic finite automata over  $\{a,b\}$  which simulates the nondeterministic finite automata

present state	next state	
	$x=a$	$x=b$
1	2,3	---
2	2	---
3	---	3

Here the start state is 1. The accepting states are 2 and 3. (5%)

- 8. Let  $A = \{1,2,3\}$ . How many different partial orderings are there on  $A$ ? How many are total orderings? (10%)
- 9. Suppose that there are two walls, located at  $N$  feet apart. A malfunctioning robot is located exactly  $n$  feet from the left wall. Every second the robot takes a step of length 1 foot, either to the right or to the left. It is equally likely to move in either direction. What is the probability that it will crash into the right wall before the left wall? (10%)