1. Given the differential equation

$$\ddot{X} + 2\omega\xi\dot{X} + \omega^2 X = f(t), \quad X(0)=0,\ \dot{X}(0)=0$$

where

$$f(t) = \begin{cases} \sin \Omega t & 0 \le t \le T \\ 0 & t \ge T \end{cases}, \quad \Omega \ne \omega^2$$

Questions:

a. What is the solution of $X(t)$ for $0 \le t \le T$? (7%)

b. What is the solution of $X(t)$ for $t \ge T$? (9%)

2. Suppose the matrix A $(=[a_{ij}]_{n\times n})$ is symmetric and whose eigenvalues $\lambda_1, \lambda_2, \ldots\ldots$ and λ_n are distinct. Let $V_1, V_2, \ldots\ldots$ and V_n be their corresponding eigenvectors.

Given the vector equation $Ax=b$

Questions:

a. What is the form of b in terms of $V_1, \ldots$ and V_n? (7%)

b. What is the solution of x in terms of $V_1, \ldots\ldots$ and V_n ? (7%)

3. Suppose that $Y_1(x)$ and $Y_2(x)$ are the solutions of the differential equations:

1) $\ddot{Y}_1 = P(x)\dot{Y}_1 + Q(x)Y_1 + R(x)$, $a \le x \le b$, $Y_1(a)=\alpha$, $\dot{Y}_1(a)=0$

2) $\ddot{Y}_2(x) = P(x)\dot{Y}_2 + Q(x)Y_2$, $a \le x \le b$, $Y_2(a)=0$, $\dot{Y}_2(a)=1$

Given the differential equation

$$\ddot{Y} = P(x)\dot{Y} + Q(x)Y + R(x),\ a \le x \le b,\ Y(a)=\alpha,\ Y(b)=\beta$$

Questions: What is the solution of $Y(x)$ in terms of $Y_1(x)$ and $Y_2(x)$? (10%)

4. Suppose that R is the region bounded by the simple closed curve C and dF/dn is the directional derivative of F in the direction of the outer normal to C. Using Green's lemma to prove

a. $\int\int_R \nabla^2 F \, dxdy = \oint_C (dF/dn)ds$ (5%)

b. $\int\int_R [(\partial G/\partial x)^2 + (\partial G/\partial y)^2]dxdy = \oint_C G(dG/dn)ds$ (5%)

if G satisfies Laplace's equation in a region R.

5. Complex variables and related questions:

a. What is the difference between Taylor series and Laurent series ? (4%)

b. Draw a simply connected domain and a doubly connected domain (4%)

c. If f(z) is analytic inside a simple closed path L and on L, except for finite number of singularities $Z_1, Z_2, \cdots, Z_n$ inside L, find the value of

$\oint_L f(z)\,dz$ and prove the result. (10%)

6. Given a partial differential equation as shown:

$$a(x,y,u)\,\partial^2 u/\partial x^2 + b(x,y,u)\,\partial^2 u/\partial x\partial y + c(x,y,u)\,\partial^2 u/\partial y^2 = f(x,y,u)$$

a. When will the PDE be linear? (4%)

b. When can the superposition principle be applied? (4%)

c. When will the solution be in the form of Fourier series, Bessel functions or Legendre functions? (12%)

7. if a function f(x) can be expressed as follows:

$$f(x)=A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} \right)$$

a. What is Gibbs phenomenon and how to overcome it ? (4%)

b. How can one compute A_n and B_n mathematically (give reasons) (4%)

c. For f(x) as shown in the figure, please indicate the approximate value of A_0 on the figure (give reasons) (4%)