1. (15%)

Consider the problem of

$$c^2 Y_{tt} = Y_{xx} + F(x,t), \quad (0 \le x \le 1,\ 0 \le t < \infty)$$

$$Y(0,t) = Y(1,t) = 0,\ Y(x,0) = f(x),\ Y_t(x,0) = 0$$

Assume the solution of $Y(x,t)$ to be the form

$$Y(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin(n\pi x)$$

What is the solution of $h_n(t)$?

2. (15%)

Find the value of $\int_0^{\infty} \frac{\cos(ax)}{x^2 + b^2}\, dx$, $a \ge 0$, $b \ge 0$.

3. (10%)

Evaluate $\int_C (6xy - 4e^x)dx + 3x^2 dy$ with C any piecewise smooth curve from (0,0) to (-2,1).

4. (10%)

Solve $\frac{d}{dt}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} g(t)$

$x_1(0)=1,\ x_2(0)=2$

5. (20%)

For a simple pendulum of length L and mass m, neglecting any frictional effects,

(a) Derive its equation of motion and discuss its equilibrium points

(b) Simplified the equation obtained in (a) to obtain the linearized equation

$\theta_{tt} + (g/L)\,\theta = 0$, where θ is the angular displacement, g is gravitational acceleration

(c) Change the equation obtained in (b) into a system of first order ordinary differential equations.

(d) Obtain the general solution of the linearized equation of motion and discuss the solution behavior.

6. (10%)

For an ordinary diffferential equation,

$$Y_{xx} + a_1 Y_x + a_2 Y = f(x), \qquad 0 \le x \le L$$

Given the following conditions, which one is an initial value problem, which one is a boundary value problem:

(a) $Y(0) = Y_x(0) = C_1$,

(b) $Y(0) = C_2$, $Y(L) = C_3$,

(c) $Y(0) = C_4$, $Y(L/2) = C_5$,

(d) $Y(L) = C_6$, $Y_x(L/2) = C_7$

Discuss briefly what is the difference between initial value problems and boundary value problems.

7. (10%)

What is the advantage of solving a problem by Laplace transform? Under what conditions one can not use Laplace transform? What is the relation (or difference) between Laplace transform and Fourier transform?

8. (10%)

Given a function f(x) as

$$f(x) = 1 - x, \quad 0 \le x \le 1$$

Compare the accuracy of the Fourier series expansion of f(x) by

a. Treat f(x) as a periodic function

b. Treat f(x) as an odd function

c. Treat f(x) as an even function

Note: It is not a required condition to find the expansion coefficient.