1. (15%)

Let $T_n[\cos(\theta)] = \cos(n\theta)$, $U_{n-1}(y) = T'(y)/n$ and $U_{-1}(y) = 0$ where $y = \cos(\theta)$. Show that

a) $T_n(y) = U_n(y) - y\,U_{n-1}(y)$, $n \geq 0$

b) $U_n(y) = 2y\,U_{n-1}(y) - U_{n-2}(y)$, $n \geq 1$

2. (10%)

Let $\{U_1, U_2, \ldots\ldots, U_n\}$ be an orthonormal basis for R^n and let $\lambda_1, \lambda_2, \ldots$ and λ_n be scalars. Define

$$A = \lambda_1 U_1 U_1^T + \ldots\ldots + \lambda_n U_n U_n^T$$

What are eigenvalues of A and their corresponding eigenvectors?

3. (15%)

Solve $\dfrac{d}{dt}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}\sin(2t)$

$X_1(0) = 1$, $X_2(0) = 2$.

4. (10%)

Let B be the bordered square matrix

$$B = \begin{bmatrix} A & U \\ V & c \end{bmatrix}$$

where U is a column vector, V is a row vector and c is a number.

What is det(B) ?

5. (1) Find the smallest equivalence relation on {a,b,c,d} which contains the relation R = {(a,b),(c,d)}. (3%)
   (2) Let R be the smallest partial ordering on {a,b,c,d} which contains {(a,b),(a,c),(a,d)}. How many elements does R have? (3%)
   (3) How many different relations are there on a 2-element set? (4%)

6. Given the definition of the partial ordering and show which (if any) of the following relations on A ={1, 2, 3} is a partial ordering? (10%)
   (a) R = {(1,1), (2,2), (3,3)}.
   (b) R = {(1,1), (2,2), (3,3), (1,2)}
   (c) R = {(1,1), (2,2), (3,3), (1,2), (2,1)}
   (d) R = {(1,1), (2,2), (3,3), (1,2), (2,3)}
   (e) R = A x A

7. A language X is defined as the following rules:
   Rule (X0): The empty string ε is in language X
   Rule (X1): If x is in language X, so is (x).
   Rule (X2): If x and y are in language X, so is (x)(y).

Now, we have the following strings: a. ((()())) b. ())(() c. (())(())(). which one is in the language X? Design a decision algorithm for language X. (10%)

8. Construct NFAs (Nondeterministic Finite Automaton)
   (a) accept the regular languages corresponding to the regular expressions:
   $xy^* + (x^* + yy)z$ (5%)
   (b) All strings which contain at most one x or at most one y, over the alphabet {x, y}. (5%)

9. According to Euler's formula, the number of vertices (V), edges (E), and faces (F) in an arbitrary connected planar map are related by the formula V + F = E + 2. Prove the formula using mathematical induction. (10%)