1. (a) Let U be a unitary matrix, i.e.,. $U^{-1} = U^*$. Prove that the eigenvalues of U have absolute value 1. (8%)
   (b) Let A be an nxn diagonalizable matrix with the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

   Prove that

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = O$$

   where O is a zero matrix. (8%)

2. Solve the differential equation

$$y'''' - 10y''' + 35y'' - 50y' + 24 = g(t)$$

   with the initial conditions

$$y(0) = y'(0) = y''(0) = y'''(0) = 0 \qquad (10\%)$$

3. Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{1+\cos\theta/4}\,d\theta$ (12%)

4. Evaluate the surface integral $\displaystyle\iint_\Sigma z\,d\sigma$

   with Σ that part of the plane x + y + z = 4 lying above the rectangle $0 \le x \le 2,\ 0 \le y \le 1$. (12%)

5. What are linear and nonlinear differential equations? (10%)
(a) Is the follow equation linear or nonlinear? Why?

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)$$

(b) What is the difference between the solution methods for these two types of equations?

6. $\displaystyle\phi(\alpha) = \int_0^1 \frac{x^\alpha + 1}{\ln x}\,dx$

   Evaluate (a) $d\phi/d\alpha$ (b) the integration, i.e. $\phi(\alpha)$. (8%)

7. Let ϕ_n be a set of functions orthogonal respect to a weighting function w(x) over a finite internal interval (a, b), which means

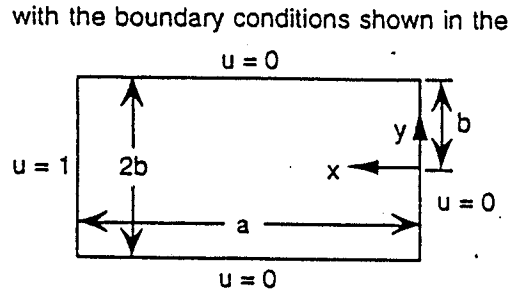
$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = \begin{cases} \neq 0 & n = m \\ = 0 & n \neq m \end{cases}$$

   If f(x) is written as $f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x)$, derive the expression of b_n and give an example of ϕ_n. (10%)

8. Solve the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

   with the boundary conditions shown in the following figure. (14%)



9. Find the solution of the following equation. (8%)

$$\frac{dy}{dx} + \frac{1}{x}y = x^3y^3$$