

1. Consider five points  $(1,0), (2,1), (3,2), (4,4)$  and  $(5,7)$  to determine the parabola  $y = ax^2 + bx + c$  of best fit in the least square sense. What are the values of  $a, b$  and  $c$ ? (15%)
2. Suppose  $x_1, x_2, \dots, x_n$  to be a basis of  $R^n$ . Determine an orthonormal basis of  $R^n$  in terms of  $x_1, x_2, \dots, x_n$ . (10%)
3. Consider the linear mapping  $T: R^n \rightarrow R^n$ . Prove that
  - (a) There exists a matrix  $A$  such that  $T(x) = Ax$ ,  $x \in R^n$ . (5%)
  - (b) If  $\text{rank}(A) < n$ , then  $T$  is not one-to-one. (5%)
4. The real symmetric matrix  $A$  has distinct eigenvalues  $\lambda_1, \dots, \lambda_n$ .
  - (a) Prove that the corresponding eigenvectors  $x_1, x_2, \dots, x_n$  form a basis of  $R^n$ . (7%)
  - (b) Express the solution of  $Ax = y$  in terms of the linear combination of  $x_1, x_2, \dots, x_n$ . (8%)
5. A scheduler of some single-CPU operating system have 3 groups of processes to schedule. They are Group A, Group B and Group C. Suppose that Group A has 3 processes, Group B has 5 processes, and Group C has 8 processes. The processes are dispatched according to the rules: (1) all the processes are of the same priority (2) once a process is dispatched, it would be kept running until it comes to a stop. In how many ways can the scheduler arrange these processes so that processes of the same group are together? (5%)
6. Consider a graph  $G$  which consists of two cycles of length  $n$  joined together by a single additional edge. Find a formula for the number of spanning trees in  $G$ . (5%)
7. Let  $A$  be a set with  $n$  elements and let  $B$  be a set with  $m$  elements, disjoint from  $A$ . If you have a graph  $G$  in which each edge joins an element of  $A$  to an element of  $B$ , what is the largest possible number of edges  $G$  can have? (5%)

8. Suppose the language L is defined by the following rules: a and ab are in L; if axb is in L then aaxbb is in L; if xa is in L then axa is in L. How many strings of length 2k are there in L?(5%)

9. Find a regular expression for the language of all strings over {x,y} which begin with an x and end with a y.(5%)

10. Find the solution to the HDE (Homogeneous Difference Equation) with the boundary conditions  $X_0 = 6, X_1 = 9, X_3 = 87$ .(5%)

$$X_n - 4X_{n-1} + X_{n-2} + 6X_{n-3} = 0$$

11. Find the generating function for the sequence (5%)  
0,0,1,0,0,1,0,0,1,0,0,1,0,0,1,.....

12. (1) Suppose that you are rolling two fair dice. Given that the sum is even, what is the probability that both die are the same? (3%)  
(2) In a poker hand of 5 cards, what is the probability that there is no pair?(3%)

13. How many different binary relations are there on an n-element set? (3%)

14. Consider a directed graph G whose length matrix is given below, (where the '-' denotes no connection):

	1	2	3	4	5	6
1	0	2	-	-	-	-
2	-	0	1	5	-	-
3	3	-	0	3	4	6
4	-	-	-	0	-	2
5	-	-	-	-	0	-
6	-	-	-	-	1	0

Please find the (1)reachability matrix (3%)(2) shortest path matrix for G.(3%)