1. The temperature at any point in space is given by

$$T = xy + yz + zx$$

(a) Find the direction cosines of the direction in which the temperature changes most rapidly with distance from the point (1, 1, 1), and determine the maximum rate of change. (6%)

(b) Find the derivative of T in the direction of the vector $3\vec{i} - 4\vec{k}$ at the point (1, 1, 1). (4%)

2. Evaluate the integral $\iint_S x^2 \, d\sigma$, where S is the portion of the plane $x+y+z=1$ inside the cylinder $x^2+y^2=1$. (6%)

3. (a) Find the solution of the set of differential equations

$$\begin{cases} 2t\dfrac{dx}{dt} = 3x - y \\ 2t\dfrac{dy}{dt} = 3y - x \end{cases}$$ (5%)

(b) Solve the differential equation $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x \ln x$ by the method of variation of parameters. (6%)

4. (a) Show that, if $F(s) = \mathcal{L}\{f(t)\}$, and if $f(t)/t$ has a transform, then

$$\int_s^\infty F(s)\,ds = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$ (4%)

(b) Find $\mathcal{L}\left\{\dfrac{e^{-at} - e^{-bt}}{t}\right\}$. $(a, b > 0)$ (4%)

(c) Evaluate $\displaystyle\int_0^\infty \frac{e^{-t} - e^{-2t}}{t}\,dt$ (4%)

5. (a) Let J_n denote the Jacobi matrix of order n;

$$J_n = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & b_{n-1} \\ 0 & \cdots & 0 & & c_{n-1} & a_n \end{bmatrix}$$ (5%)

Show that $|J_n| = a_n|J_{n-1}| - b_{n-1}c_{n-1}|J_{n-2}|$ $(n \geq 3)$

(b) Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$

Find adj A, $|A|$, A^{-1} (6%)

6. Find the two half-range expansions of the function

$$f(x) = x^2 \quad (0< x< L) \quad .$$ (10%)

7. What are the corresponding eigenvalue problems of the following problems:

$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, subjected to the conditions: (c^2 is a constant)

(a) $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = f(x)$.

(b) $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$, $u(x,0) = f(x)$.

And solve the eigenvalue problems. (20%)

8. Answer the following questions: (20%)

(1) Wronskian = $W(y_1, y_2, \ldots, y_n)$ = ? for linear dependence and independence of functions $y_1, y_2, \ldots, y_n$.

(2) What's the radius of convergence of the series $\sum_{m=0}^{\infty} x^m/m!$?

(3) Legendre polynomial of degree n, $P_n(1)$ = ?

(4) Gamma function, $\Gamma(\alpha+1)$= ? for $\alpha>0$.

(5) An orthonormal set $g_1, g_2, \ldots$ on an interval $a \le x \le b$, (g_m, g_n) = ? $m=1,2,\ldots;n=1,2,\ldots$

(6) $L^{-1}[\ 1\ ]$ = ?

(7) Does $1*f = f$ in general?

(8) Does $\overline{A}\overline{B} = \overline{0}$ imply $\overline{A} = \overline{0}$ or $\overline{B} = \overline{0}$?

(9) $\text{erf}(\infty)$= ?

(10) Jacobian = $J = \frac{\partial(x,y)}{\partial(r,\theta)}$ = ? where x,y, are rectangular coordinates and r,θ, are polar coordinates.