1. Solve the following ordinary differential equations:

   (a) $xy'' + y' = y'^2$ (5%)

   (b) $\begin{cases} x' = -2x + y \\ y' = -4x + 3y + 10\cos t \end{cases}$ (5%)

   (c) $y'' - 2y' + y = e^x + x$ (5%)

   (d) $(x^2D^2 - 2xD + 2)y = x^3\cos x$ (5%)

2. Answer the following questions: (20%)

   (1) Wronskian = $W(y_1, y_2, \ldots, y_n)$ = ? for linear dependence and independence of functions $y_1, y_2, \ldots, y_n$.

   (2) What's the radius of convergence of the series $\sum_{m=0}^{\infty} x^m/m!$?

   (3) Legendre polynomial of degree n, $P_n(1)$ = ?

   (4) Gamma function, $\Gamma(\alpha+1)$ = ? for $\alpha > 0$.

   (5) An orthonormal set $g_1, g_2, \ldots$ on an interval $a \le x \le b$, (g_m, g_n) = ? m=1,2,..;n=1,2,...

   (6) $L^{-1}[\ 1\ ]$ = ?

   (7) Does 1*f = f in general?

   (8) Does $\bar{A}\bar{B} = \bar{0}$ imply $\bar{A} = \bar{0}$ or $\bar{B} = \bar{0}$?

   (9) $\bar{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\bar{A}^{-1}$ = ?

   (10) Jacobian = $J = \dfrac{\partial(x,y)}{\partial(r,\theta)}$ = ? where x,y, are rectangular coordinates and r,θ, are polar coordinates.

3. Mathematical problems are most easily solved in nondimensional form. Illustrate the procedure for turning the model equation into nondimensional form by considering the steady-state reaction and diffusion in a slab, which can be modelled as:

   $$D\frac{d^2c}{dx^2} - kc^2 = 0, \quad -D\frac{dc}{dx} = 0, \ x=0; \quad c = c_0, \ x=L \ ,$$

   where D, k are the diffusion coefficient and thermal conductivity, respectively. Don't try to solve the problem. (10%)

4. A tank contains 2 m^3 of water. A stream of brine containing 20 kg/m^3 of salt is fed into the tank at a rate of 0.02 m^3/s. Liquid flows from the tank at a rate of 0.01 m^3/s. If the tank is well agitated, what's the salt concentration in the tank when the tank contains 4 m^3 of brine? (10%)

5. The reaction rate constant for the decomposition of a substituted dibasic acid has been determined at various temperatures as follows:

T(°C)	50.0	70.1	89.4	101.0
$k\times10^4(h^{-1})$	1.08	7.34	45.4	138

   How can you determine the activation energy (E) in the equation $k = A\, e^{-E/RT}$, where T is measured in degrees Kelvin, by using the method of least squares? (10%)

6. Find the two half-range expansions of the function

   $$f(x) = x^2 \quad (0 < x < L) \ .$$ (10%)

7. What are the corresponding eigenvalue problems of the following problems:

   $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$, subjected to the conditions:

   (a) $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = f(x)$.

   (b) $\dfrac{\partial u}{\partial x}(0,t) = 0$, $\dfrac{\partial u}{\partial x}(L,t) = 0$, $u(x,0) = f(x)$.

   And solve the eigenvalue problems. (20%)