(甲)

1 (a) Find the function $S(b,t) = \mathcal{L}^{-1}\left\{\dfrac{e^{-bs}}{s(1-e^{-bs})}\right\}$ and plot S(b,t) versus t. (5%)

(b) Prove that $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \dfrac{1}{s}\mathcal{L}\{f(t)\}$. (7%)

2 (a) Prove that $\dfrac{dA^{-1}(t)}{dt} = -A^{-1}(t)\dfrac{dA(t)}{dt}A^{-1}(t)$, where A(t) is an n × n nonsingular matrix. (5%)

(b) Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 3 \\ -2 & 1 & 1 \\ 4 & 2 & 5 \end{bmatrix}$. Find $\lambda_1+\lambda_2+\lambda_3$ and $\lambda_1\lambda_2\lambda_3$. (5%)

3 (a) How do you use the Simpson's rule of numerical integration to evaluate the integral $\int_a^b f(t)\,dt$ where a and b are constants. (5%)

(b) How do you use the Newton's iteration method to find a real root of f(x) = 0. (5%)

4 Given the partial differential equation

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

with the initial and boundary conditions:

$$\begin{aligned} &y(x, t) = 0 && \text{for } t \le 0 \text{ and all } x, \\ &y(0, t) = 1 && \text{for all } t > 0, \\ &y(\infty, t) = 0 && \text{for all } t > 0. \end{aligned}$$

Show that with an appropriate value of α, the transformation $\eta = y\,t^{\alpha}$ can reduce the partial differential equation to an ordinary one in terms of the new independent variable η. Also, find $y(\eta)$. (15%)

5 Let

$$f(i) = \sum_{k=0}^{N-1} F^*(k)\exp\left[-j\frac{2\pi ik}{N}\right]$$

and

$$h(i) = \sum_{k=0}^{N-1} H(k)\left\{\cos\frac{2\pi ik}{N} + \sin\frac{2\pi ik}{N}\right\}$$

where * denotes complex conjugate, $j = \sqrt{-1}$, and i = 0, 1, ..., N-1. Prove that if H(k) = Re{A(k)} - Im{A(k)}, then h(i) = Re{f(i)} + Im{f(N-i)}, where Re{C} ≡ real part of the complex C and Im{C} ≡ imaginary part of C. (10%)

6 $y_1 = e^{-2x}$ is a solution of the equation $x y'' + (2x-1)y' - 2y = 0$ $(x > 0)$. Find a second, linearly independent solution to the above differential equation. (10%)

7 (a) Is $\vec{F}(x, y, z) = -\frac{y}{z}\vec{i} - \frac{x}{z}\vec{j} + \frac{xy}{z^2}\vec{k}$ conservative in the region $z > 0$? Why? (5%)

(b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is a piecewise smooth curve joining the points (1,1,1) and (2,-1,3) and not crossing the xy-plane. (5%)

8 (a) Transform the integral $\int\int_R e^{(x-y)/(x+y)}\,dxdy$, to an integral in uv-plane by using the transformation $u = x - y$ and $v = x + y$, where R is the region in the first quadrant between the lines $x + y = 2$ and $x + y = 3$. (7%)

(c) Evaluate the above integral. (6%)

9 Find the first four terms of Maclaurin series of the function

$$f(x) = \ln\left\{\frac{1+x}{1-x}\right\} \qquad \text{for } |x| < 1. \quad (10\%)$$

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