1 (10%) Determine the eigenvalues and eigenfunctions of the differential equation

$$x\,y'' + y' + \lambda\,x\,y = 0$$

with $y(0) = \text{finite}\ , \quad y(a) = 0$

2 The final–value theorem states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,\mathcal{L}\{f(t)\}$$

where $\mathcal{L}\{f(t)\}$ is the Laplace Transform of f(t).

(a) (5%) Is the above theorem valid for
(i) $\mathcal{L}\{f(t)\} = \dfrac{1}{s-1}$, (ii) $\mathcal{L}\{f(t)\} = \dfrac{1}{s+1}$?

(b) (5%) Give the reason(s) why the final–value theorem fails in some cases. In addition, state the limitation(s) of the theorem.

3 (a) (5%) Evaluate the integral $\iint_S \vec{n}\cdot\nabla\times\vec{v}\ dA$ over the part of the unit sphere $x^2 + y^2 + z^2 = 1$ above the xy plane, where $\vec{v} = y\,\vec{i}$.

(b) (5%) If $\vec{p}$ is the position vector, show that

$$\oiint_S \vec{p}\cdot\vec{n}\ dA = 3\,V$$

where V is the volume enclosed by a closed smooth surface S, and $\vec{n}$ is the normal outward vector on S.

4 (a) (8%) Please find the Fourier Integral representation of the function

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

(b) (7%) Evaluate $\displaystyle\int_0^\infty \left\{ \frac{x\cos x - \sin x}{x^3} \right\} \cos\frac{x}{2}\, dx$

5 (10%) It is known that $y_1 = x$ is a solution of the equation

$$(x^2 - 1)\,x^2\,y'' - (x^2 + 1)\,x\,y' + (x^2 + 1)\,y = 0$$

Find a second linearly independent solution to the above equation.

6 Solve the following equation

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x\,t$$

with

$$u(x, 0) = 0, \qquad x \geq 0$$
$$u(0, t) = 0, \qquad t \geq 0$$

(a) (10%) by the Laplace transformation.

(b) (10%) by separating variables.

7 (15%) Find the solution to the following set of differential equations by the method of matrix diagonalization.

$$\mathbf{y'} = \begin{bmatrix} -4 & -6 \\ 1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 9e^{-3t} \\ -5e^{-3t} \end{bmatrix}$$

with $\mathbf{y}(0) = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$.

8 (10%) In solving a differential equation by the finite difference method, one replaces the differential operator by a difference expression. It will be convenient to consider a constant increment in the independent variable x such that $\Delta x = x_{n+1} - x_n$ and $n\,\Delta x = x_n$. Also let $\psi(x_n)$, the dependent variable in the differential equation, be denoted by ψ_n. The following operators are defined as

forward difference $\quad \Delta\psi_n = \psi_{n+1} - \psi_n$
backward difference $\quad \nabla\psi_n = \psi_n - \psi_{n-1}$
central difference $\quad \delta\psi_n = \psi_{n+1/2} - \psi_{n-1/2}$

Show that $\quad \nabla\Delta\psi_n = \Delta\nabla\psi_n = \delta^2\psi_n$