1 (10%) How can you determine the half–life of a radioactive substance from two measurements $y_1 = y(t_1)$ and $y_2 = y(t_2)$ of the amounts present at times $t_1$ and $t_2$?

2 (10%) It is known that $y_1 = x$ is a solution of the equation

$$x\,y'' + x\,y' - y = 0$$

Find a second linearly independent solution to the above equation.

3 (a) (5%) Determine the coefficients in the representation

$$1 = \sum_{n=1}^{\infty} A_n \sin nx \qquad (0 < x < \pi)$$

(b) (5%) If the expansion $f(x) = \sum_{n=1}^{\infty} A_n \sin nx$ is valid in $(0,\pi)$, show that

$$\int_0^{\pi} [f(x)]^2\,dx = \frac{\pi}{2} \sum_{n=1}^{\infty} A_n^2$$

(c) (5%) Show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

4 (10%) The linearization of a function $f: \Re^n \to \Re$ at $x_0$ is the first degree Taylor polynomial that approximates that function at $x_0$. Please find the linearization of $f = \sqrt{x+y}$ at $(x, y)_0 = (2, 2)$.

5 (10%) In a temperature field, heat flows in the direction of maximum decrease of temperature T and curves of constant temperature, T = constant, are called isotherms. Find the direction of heat flow at P(1,1) if $T = y^2 + 2xy - x^2$. In addition, please sketch the curves of heat flow and isotherms on your answer sheet.

6 (15%) Consider a model the population of two species governed by the system

$$x_1'(t) = -\frac{1}{2}x_1(t) + x_2(t)$$

$$x_2'(t) = \frac{1}{4}x_1(t) - \frac{1}{2}x_2(t)$$

Suppose that $x_1(0) = 200$, $x_2(0) = 500$. Determine the population of each species for $t > 0$.

7 (10%) Solve the initial value problem

$$y'' + 3y' + 2y = r(t), \quad y(0) = 0, \quad y'(0) = 0$$

with $r(t) = 1$ if $0 < t < 1$ and 0 otherwise.

8 (5%) Find $(\delta * f)(t)$, where $\delta$ is Dirac's delta function and "*" means convolution.

9 (10%) Find the arc length and area of one arch of the cycloid:

$$\vec{r} = a(t - \sin t)\vec{i} + a(1 - \cos t)\vec{j}, \quad 0 \le t \le 2\pi$$

10 (5%) Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 5 & 10 \\ 4 & -1 \end{bmatrix}$$