1 (8%) Solve for the general solution

$$x\,y' + y + 1 = 0 \qquad (\,' = \frac{d}{dx}\,)$$

2 (10%) Prove $J_{-n}(x) = (-1)^n J_n(x)$ where

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

3 (10%) Find the Laplace transform of the function

$e^{-t}\,u(t-2)$, where $u(t)$ is the unit step function.

4 (10%) Find the eigenvalues and eigenvectors of the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5 (10%) Find the Fourier Transform

$$f(x) = \begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (a > 0)$$

6 (8%) Determine the rank of the following matrix

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 8 & 3 & 1 \\ 1 & 0 & 3 & 1 \end{bmatrix}$$

7 (12%) The time rate of change of a population $y(t)$ can be described by the logistic law:

$$\frac{dy}{dt} = a\,y - b\,y^2 \qquad (a > 0,\ b > 0)$$

where the "braking term" $-by^2$ has the effect that the population cannot grow indefinitely. Solve this Bernoulli equation (Hint: let $u = y^{-1}$). What is the limit of $y(t)$ as $t \to \infty$?

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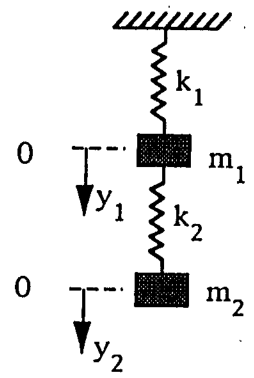
8 (14%) Set up the model of the undamped vibrating mechanical system in the following figure for general m_1 (mass), m_2, k_1 (spring modulus), and k_2. Find the displacements of functions of time, $y_1(t)$ and $y_2(t)$, of the masses from their positions of static equilibrium ($y_1 = 0$, $y_2 = 0$) when the whole system is at rest. Here $m_1 = 1.5$, $m_2 = 2$, $k_1 = 4.5$, and $k_2 = 6$.



9 (10%) If you know $f(t) = \mathcal{L}^{-1}\{F(s)\}$, how would you find $\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\}$? In addition, determine $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2(s+1)}\right\}$

10 (8%) For what c are the planes $x + y + z = 1$ and $2x + cy + 7z = 0$ orthogonal?