1 (8%) Evaluate the flux of water through the surface S: which is the portion of the plane $x + y + z = 2$ (unit: meter) in the first octant, when the velocity vector is $F = 1 + 2k$, speed being measured in meter/sec.

2 (12%) Solve the nonhomogeneous equation:

$$y'' - 4y' + 4y = xe^{2x} + x$$

3 (10%) Solve $y(t) = 1 + \int_0^t y(\tau)\, d\tau$

4 (9%) Determine the rank, eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5 (10%) Find the f(t) if it Laplace transform $\mathcal{L}(f)$ equals

$$\frac{s + 2}{(s + 1)^2}$$

6 (10%) If $y_1(x)$ is a solution of $y''' + p_2(x)\, y'' + p_1(x)\, y' + p_0(x)\, y = 0$, show that another solution can be found by letting $y_2(x) = u(x)\, y_1(x)$ with $u(x) = \int z(x)\, dx$ where z is obtained from

$$y_1 z'' + (3y_1' + p_2 y_1) z' + (3y_1'' + 2p_2 y_1' + p_1 y_1) z = 0$$

7 (8%) Find the general solution of the homogeneous linear system

$$y'(t) = Ay = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y \quad \text{where } y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

8 (10%) Given that $xe^y - y^2 - z^2 \sin z = 0$, find

$$\frac{\partial y}{\partial x}$$

9 (8%) A physical phenomenon is described by the quantities P (pressure), ℓ (length), m (mass), t (time), and ρ (density). If there is a physical law

$$f(P, \ell, m, t, \rho) = 0$$

relating these quantities, show that there is an equivalent physical law of the form

$$G\left(\frac{P\ell^3}{m}, \frac{t^6 p^3}{m^2 \rho}\right) = 0$$

10 (15%) The equation

$$\frac{\partial C}{\partial t} = D \nabla^2 C \qquad (1)$$

appears in a number of problems in sciences and engineering. With suitable assumptions and corresponding initial and boundary conditions, it can describe:

(i) heating and cooling (spheres, rods, etc.),
(ii) mass transfer in films, fibers, etc.,
(iii) diffusion-controlled reactions (in solution or in a catalyst pellet),
(iv) start-up flow of a Newtonian fluid,
... and many others.

Select one of the above examples and in the context of the example, explain:

(a) Where does equation (1) come from?
(b) Explain the physics of your problem and give suitable initial and boundary conditions.
(c) How to solve your problem?