

※ 考生請注意：本試題不可使用計算機。請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (20%)

(1) (10%) Find a homogeneous ODE for the given basis: $x^2, x^{-3}\cos(2\ln x), x^{-3}\sin(2\ln x)$.

(2) (10%) Find the general solution of $y'' - 4y' + 4y = \frac{e^{2x}}{x-1}$.

2. (20%)

Given a matrix $\mathbf{X} = \begin{bmatrix} 3-p & -3 & 0 \\ 0 & 2-p & 0 \\ 0 & -1 & 1-p \end{bmatrix}$, where p is a constant.

(1) (10%) Determine the relationship between the rank of \mathbf{X} and the value of p .

(2) (10%) Determine the \mathbf{X}^{-1} when $p = 4$.

3. (20%)

(1) (10%) Let f and \mathbf{v} be scalar and vector functions, respectively. Show that $\text{curl}(f\mathbf{v}) = (\text{grad } f) \times \mathbf{v} + f \text{curl } \mathbf{v}$.

(2) (10%) Let f and g be two scalar functions. Show that $\nabla^2(fg) = f\nabla^2g + 2\nabla f \cdot \nabla g + g\nabla^2f$.

4. (20%)

(1) (10%) Determine the flux of water through the surface of $x^2 + y^2 + z^2 = 4$ ($z \geq 0$) if the velocity vector is $\mathbf{V} = [x, x, z]$.

(2) (10%) Verify Stokes's theorem for the given $\mathbf{F} = [y^3, -x^3, 0]$ and $S: x^2 + y^2 \leq 4$ and $z = 3$.

5. (20%)

Using the Laplace transform to solve the linear system.

$$\begin{cases} y_1'' - y_2' = 2\cos 2t \\ y_2'' + y_1' = -2\sin 2t \end{cases}, y_1(0) = -1, y_2(0) = 1, y_1'(0) = 1, y_2'(0) = 2.$$