1. As shown in Fig. 1, a cylindrical tank 1.5 meters high stands on its circular base of diameter 1.0 meter and is initially filled with water. At the bottom of the tank there is a hole of diameter 1.0cm, which is opened at some instant. The Torricelli's law states that the velocity with which water issues from an orifice is $v = 0.6\sqrt{2gh}$, where $g = 980 cm/\sec^2$ and h is the instantaneous height of the water above the orifice. Find (a) the height h(t) of the water in the tank; (b) the time at which the tank is empty. (10%)

2. The Legendre polynomial of degree n is expressed as

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

where M = n/2, whichever is an integer. Find
(a) $P_0(x)$, $P_1(x)$ and $P_2(x)$; (6%)

(b) Represent the polynomial in terms of the Legendre polynomial $3x^2 + x = a_0P_0(x) + a_1P_1(x) + a_2P_2(x)$, find constants a_0 a_1 and a_2.(6%)

3. Solve the initial value problem

$$y'' + 2y = r(t), \quad y(0) = 0,\ y'(0) = 0,$$

where r(t) =1 if 0<t<1 and 0 otherwise. (10%)

4. Find the acceleration $a(t)$ of a projectile P moving along a meridian M of a rotating sphere with angular speed γ relative to the sphere, which also rotates with angular speed ω and radius R shown in Fig. 2. (13%)

5. Evaluate the double integral

$\iint_R (x^2 + y^2)dxdy$ where R is the region shown in Fig. 3. (10%)

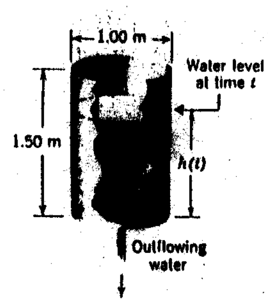

Fig. 1

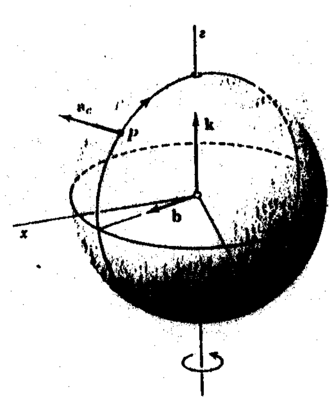

Fig. 2

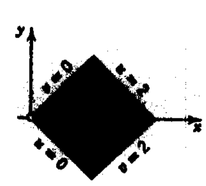

Fig. 3

6. Find the Fourier integral representation of the function shown in Fig. 4 and prove

$$\int_0^\infty \frac{\cos wx \ \sin wx}{w}\, dw = \begin{cases} \pi/2 & \text{if } 0 \le x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases} \qquad (13\%)$$

7. (a) Derive the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho}$$

where u = u(x,t) is the deflection of a vibrating string (Fig. 5) at any point and any time. (10%)

(b) Solve the wave equation with initial conditions: (10%)

$$u(x,0) = f(x) = \frac{1}{1+8x^2}$$
$$u_t(x,0) = g(x), \quad g(x) = 0$$

8. Integrate

$$g(z) = \frac{z^2+2}{z^2-1}$$

in the counterclockwise sense around a circle of radius 1 with center atthe point (a) z =1, (b) z = 1+i and (c) z = i. (12%)

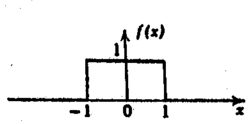

Fig. 4

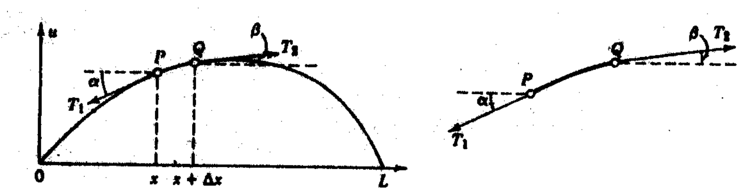

Fig. 5