國立成功大學76學年度航太所入學考試(工程數學 試題) 共二頁 第1頁

(甲.乙.丙)

1. Consider a function f defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

a) Is $f(x)$ continuous over $(-\infty, \infty)$? Give your argument. (2 %)
b) Find the derivative of $f(x)$ at $x \neq 0$. (2 %)
c) Show that $f'(0)$ exists and that $f'(0) = 0$. (6 %)
d) Does $f'(x)$ exist for all x? Give your argument. (2 %)
e) Is $f'(x)$ continuous at $x = 0$? Give your argument. (3 %)

2. Use the *Laplace transform* to find the solution of the simultaneous equations

$$\begin{aligned} x_1' &= x_1 + x_2 + x_3, \\ x_2' &= 2x_1 + x_2 - x_3, \\ x_3' &= -8x_1 - 5x_2 - 3x_3, \end{aligned}$$

which satisfies the initial conditions

$$x_1(0) = 1, \quad x_2(0) = 0, \quad \text{and } x_3(0) = 0.$$

Here $'$ denotes the differential symbol with respect to time, i.e., $x' = \frac{dx}{dt}$. (15 %)

3. The Fourier transform of a function $f(x)$ is a function $F(k)$ depeding on frequency:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

a) If

$$\begin{aligned} f(x) &= \text{decaying pulse} \\ &= \begin{cases} e^{-ax}, & x > 0; \\ 0, & x < 0, \end{cases} \end{aligned}$$

Find $F(k)$. (5 %)

b) If

$$\begin{aligned} f(x) &= \text{odd decaying pulse} \\ &= \begin{cases} e^{-ax}, & x > 0; \\ -e^{ax}, & x < 0, \end{cases} \end{aligned}$$

Find $F(k)$. (5 %)

c) Show schematically the profile of $f(x)$ and $F(k)$ for the above two cases. (5 %)

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4. One can define a coordinate transformation between (x, y) and (r, θ) by the following relations

$$x = r\cos\theta, \qquad y = r\sin\theta;$$
$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}.$$

Similarly, can one define a coordinate transformation between (x, y) and $(z, \bar{z})$ by the following relations?

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z});$$
$$z = x + iy, \qquad \bar{z} = x - iy$$

Explain why. (5 %)

5. Solve the following *linear algebra* problems:

a) State the condition(s) under which a matrix A with repeated eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda$ is diagonalizable? (7 %)

b) If $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$, find e^{At}! (8 %)

6. Formulate each of the following problems and solve it. Make your assumptions if necessary.

(a). By Newton's law, the rate of cooling of some body in air is proportional to the difference between the temperature of the body and the temperature of the air. If the temperature of the air is $20°C$ and boiling water cools in 20 minutes to $60°C$, *how long* will it take for the water to drop in temperature to $30°C$? (10 %)

(b). A *spherical* raindrop evaporates at a rate proportional to its surface area. Find a formula for its volume V as a function of time t. At $t = 0$, the volume of the raindrop is $V = V_o$. (10 %)

Hint: In the final expression, you *don't* even need the radius and surface area of the raindrop!!!

7. In the circular cylindrical coordinate system, a vector $\vec{f}$ is defined as

$$\vec{f} = \vec{e}_r(z\sin\theta + 2r\cos^2\theta) + \vec{e}_\theta(z\cos\theta - 2r\cos\theta\sin\theta) + \vec{e}_z r\sin\theta$$

where $\vec{e}_i$ is the unit vector in the i-direction.

a) Evaluate the surface integral

$$\int_S \nabla \times \vec{f} \cdot \vec{N}\, dS$$

where S is the upper and side surface of the cylinder as shown and $\vec{N}$ is the outdraw normal of the surface. (7 %)

b) Evaluate the volume integral

$$\int_V \nabla \cdot \vec{f}\, dV$$

where V is the volume of the cylinder as shown. (8 %)