航空太空工程研究所

1. Prove the Caley-Hamilton theorem.
   Namely for the polynomial $Q(S) = \det|S \cdot I - A|$
   where A is a $n \times n$ square matrix, I is a $n \times n$ idendity matrix, S is a scalar variable and $\det|\cdot|$ denote the determinant of the matrix, Prove that $Q(A) = 0$. (10%)

2. (a) Show that $r=2$ is a triple root of the characteristic equation for the system

$$\underset{\sim}{X}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \underset{\sim}{X}$$

   where $\underset{\sim}{X}$ is a column vector, prime "$'$" denotes differentiation with respect to the independent variable t. (3%)

   (b) Find three linearly independent solutions of this system (6%)

   (c) Prove that the solutions obtained are linearly independent by checking the Wronskian of the system (6%)

3. Solve for $y = y(x)$

   (a) $yy' + 4y' + (1+x^2)e^{1+x^2} = 0$ (5%)

   (b) $y'' + ay = 2\sin^2 bx + 5\cos cx$ (5%)

航空太空工程研究所

4. A complex function h is given by

$$h(z) = \phi(x, y) + i\,\psi(x, y)$$

where $z = x + iy$, $i = \sqrt{-1}$, x, y are real variables and ϕ, ψ are real functions.

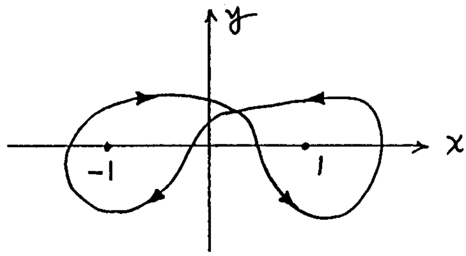
(a) What are the conditions one can say that h is analytic? (3%)

(b) If h is analytic and $\nabla h \neq 0$, determine the angles formed between lines which ϕ = constant and lines which ψ = constant in the complex z-plane. (5%)

(c) If h is analytic inside $|z| \leq 2$, evaluate

$$\int_C \frac{h(z)}{z^2 - 1}\, dz$$

where C is shown in the figure



(7%)

5. Let $\bar{f}(s) = \int_0^\infty f(t) e^{-st} dt \equiv L(f)$ be the Laplace transform of $f(t)$.

(a) What are the conditions $f(t)$ must satisfy in order for existing the Laplace transform $\bar{f}(s)$? (3%)

(b) The Laplace convolution of $f(t)$ and $g(t)$ is defined as

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Show that $L(f*g) = \bar{f}(s)\,\bar{g}(s)$ (5%)

(c) Use the Laplace transform to solve the following problem.

$$t\ddot{x} + \dot{x} + t x = 0$$

$$x(0) = 1, \quad \dot{x}(0) = 0$$ (7%)

航空太空工程研究所

6. (a) Solve the following problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$t=0$, $0 \le x < +\infty$, $u=0$

$t>0$, $x=0$, $u=1$ (7%)

(b) If the initial and boundary conditions of (a) becomes

$t=0$ $0 \le x \le 1$ $u=0$

$t>0$ $x=0$ $u=1$

What will be the differences in solution method and solution type between (a) and (b) ? Why ?

(Note: You do not have to solve the equation in (b)) (8%)

7. Find the solution $U(r,\theta)$ of Laplace's equation

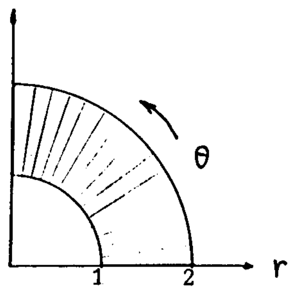
$$U_{rr} + \frac{U_r}{r} + \frac{U_{\theta\theta}}{r^2} = 0$$

inside a 90° sector of a circular annulus,

$1 < r < 2$, $0 < \theta < \frac{\pi}{2}$, also satisfying the boundary conditions

$U(1,\theta) = \sin(2\theta)$, $U(2,\theta) = 0$, $0 \le \theta \le \frac{\pi}{2}$

$U(r,0) = 0$. $U(r,\frac{\pi}{2}) = 0$, $1 \le r \le 2$



(10%)

8. (a) Solve the integral $\iint_{r \ge 1} \cdots \int \frac{dx_1\, dx_2 \cdots dx_n}{r^\alpha}$

where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$ and $\alpha > n$. (5%)

(b) Show that when $n=3$, it becomes $\frac{-4\pi}{3-\alpha}$ (5%)