(甲.乙.丙組)

1. (12 points)
   a). Expand the following function in a *Fourier series* of period equal to the indicated interval of representation:
   $$f(x) = a + bx, \qquad c < x < c + P$$
   where a, b, c, and P are constants. (6 points)
   b). Using the result of a) and by evaluating the integral
   $$\int_c^{c+P} [f(x)]^2\, dx$$
   prove that
   $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. (10 points)
   a). Mathematically show whether the integral
   $$I = \int_{(1,2)}^{(3,4)} [(6xy^2 - y^3)\, dx + (6x^2y - 3xy^2)\, dy]$$
   is independent of path of integration or not. (5 points)
   b). If so, evaluate the integral I. (5 points)

3. (12 points)
   Find the only solution of $\nabla^2 u = 0$ which depends only on $r = \sqrt{x^2 + y^2 + z^2}$.

4. (12 points)
   Find the general solution of the following differential equation
   $$yy'' - (y')^2 = y^2 \log y - x^2 y^2.$$

5. (14 points)
   a). What (acute) angle does $\vec{A} = 2\hat{i} - \hat{k}$ make with the normal to the plane containing the vectors $\vec{B} = \hat{j} + \hat{k}$ and $\vec{C} = \hat{i} - \hat{j} - \hat{k}$? (5 points)
   b). For $\vec{V} = y\hat{i} - x\hat{j} + (3z+1)\hat{k}$, evaluate
   $$I = \int_S \hat{n} \times \nabla \times \vec{V}\, d\sigma,$$
   where S is the hemispherical cap specified by $z = (a^2 - x^2 - y^2)^{1/2}$, and $\hat{n}$ is the unit vector normal to S. (5 points)
   c). Compute $(\vec{a} \times \nabla) \cdot \vec{r}$, where $\vec{r}$ is a position vector, $\vec{a} = (a_1, a_2, a_3)$, and ∇ is the gradient operator. (4 points)

(甲、乙、丙組)

6. **(12 points)**

   a). What condition(s) should the components of $\mathbf{c}$ satisfy if $A\mathbf{x} = \mathbf{c}$ is to possess a solution, where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & 6 \end{pmatrix}.$$

   b). Reduce the quadratic form

$$F(x_1, x_2, x_3) = x_1^2 - 8x_1x_2 + 4x_1x_3$$

   to canonical form. Is this quadratic form *positive definitive*? (6 points)

7. **(16 points)**

   a). Consider $w = f(z)$ which defines a mapping of the z-plane onto the w-plane. Under what conditions will the mapping be a conformal mapping? (3 points)

   b). $\phi(x, y)$ is a solution of the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

   prove that

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0 \qquad (*)$$

   if $w = f(z)$, where $w = u + iv$ and $z = x + iy$, is a conformal mapping. (7 points)

   c). State your reasons that

$$f(w) = iT_o + \frac{1}{\pi}(T_1 - T_o)\ln(w - u_o)$$

   is analytic everywhere in the upper half plane, where T_o, T_1, and u_o are real numbers. (3 points)

   d). State your reasons that

$$g(w) = T_o + \frac{1}{\pi}(T_1 - T_o)\arg(w - u_o)$$

   satisfy eq. $(*)$, where $\arg z$ means the argument of z (i.e. angle θ of z). (3 points)

8. **(12 points)**

   Water containing 2 oz of pollutant/gas flows through a treatment tank at a rate of 500 gal/min. In the tank, the treatment removes 2 % of the pollutant per minute, and the water is throughly stirred. The tank holds 10,000 gal of water. On the day the treatment plant opens, the tank is filled with fresh pure water. Find the function which gives the concentration of pollutant in the outflow.