1. **(12 points)**
Evaluate the following integral

$$\int_C \frac{z^2+\frac{1}{3}}{z^3-z}\,dz, \qquad C: \left|z-\frac{1}{2}\right| = 1 \text{ (Clockwise)}$$

2. **(12 points)**
The gamma function $\Gamma(x)$ is defined as

$$\Gamma(x) \equiv \int_0^\infty t^{x-1}e^{-t}\,dt, \qquad \text{for} \quad x > 0.$$

a). Deduce and *explain*

$$\Gamma(x) = (x-1)\Gamma(x-1) \qquad \text{for} \quad x > 1$$

and

$$\Gamma(n) = (n-1)! \qquad \text{(4 points)}$$

b). From "a)", define $\Gamma(x)$ for $x < 0$. Compute $\Gamma(1/2)$ and then $\Gamma(-1/2)$. (4 points)

c). What are the singular points for $\Gamma(x)$, $-\infty < x < \infty$. Explain why. (2 points)

d). Evaluate

$$I = \int_0^\infty e^{-x^4}\,dx. \qquad \text{(2 points)}$$

3. **(12 points)**
The vibration of a membrane is governed by the following equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial t^2}$$

a). Find the lowest mode of the square membrane with vertices at $(0,0)$, $(\pi,0)$, (π,π) and $(0,\pi)$. (4 points)

b). Use "a)", find the lowest frequency of vibration of a triangular membrane with vertices at $(0,0)$, $(\pi,0)$, (π,π).
Hint: Consider the lowest mode of the square membrane that has $y = x$ as a nodal line (*i. e.*, $w = 0$ along $y = x$). (8 points)

4. **(12 points)**
Consider a constant matrix A, $A \in R^{n\times n}$, $Ax = b$ where $x \in R^{n\times 1}$, $b \in R^{n\times 1}$

a). For any b, $b \neq 0$, if solution x exists, what condition should A have.

b). For $b = 0$, if solution x exists, what condition should A have.

c). Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, find the eigenvalues and eigenvectors.

5. **(12 points)**

$f(t)$ is a periodic function of period T and can be represented by a Fourier series

$$f(t) = a_o + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi}{T}t + b_n \sin\frac{2n\pi}{T}t\right)$$

a). Derive the expressions for the Fourier coefficients a_o, a_n, b_n, $n = 1, 2, \cdots$.

b). Using $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$, where $i = \sqrt{-1}$, show that $f(t)$ may be written as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2n\pi t/T}$$

and give the expression for the complex Fourier coefficients c_n.

c). Using the result from "a)", find the Fourier series of

$$f(t) = |t|, \quad -2 < t < 2, \qquad f(t+T) = f(t) \quad \text{and} \quad T = 4.$$

6. **(14 points)**

a). Find the Fourier transforms $F(k)$ of the following functions $f(x)$:

i). $f(x) =$ delta function $\delta(x)$

ii). $f(x) =$ decaying pulse $= \begin{cases} e^{-ax}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$

b). Solve the differential equation

$$\frac{du}{dx} + au = h(x)$$

by taking Fourier transforms to find $U(k)$. What is the solution u if $h(x) = \delta(x)$?

7. **(12 points)**

Solve the differential equation

$$y''' - \left(\frac{3}{x^2}\right)y' + \left(\frac{3}{x^3}\right)y = \ln x, \qquad {}' \equiv \frac{d}{dx}$$

8. **(14 points)**

a). Find a unit vector perpendicular to the surface

$$x^2 + y^2 + z^2 = 3$$

at the point $(1, 1, 1)$.

b). Derive the equation of the plane tangent to the surface at $(1, 1, 1)$.