1. (14%) Find the principal value of $(1+2i)^{1-2i}$

2. (14%) If $\vec{F} = 4xz\,\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$, evaluate $\iint_S \vec{F}\cdot\vec{n}\,dS$ where S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

3. (14%) Find a general solution of the following differential equations

$$(a)\quad y'' - 2y' + y = e^x + x$$

$$(b)\quad y'' + 2y' + 2y = e^{-x}/\cos^3 x$$

4. a) (10%) Using the method of separation of variables to find particular solutions:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} - u = 0 & \text{for} \quad 0 < x < 1, \quad t > 0 \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & \\ u(0,t) = u(1,t) = 0 & \end{cases}$$

b) (5%) Show that the problem

$$\begin{cases} \dfrac{\partial}{\partial x}\left(e^x \dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(e^y \dfrac{\partial u}{\partial y}\right) = 0 & \text{for} \quad x^2 + y^2 < 1 \\ u = x^2 \quad \text{for} \quad x^2 + y^2 = 1 & \end{cases}$$

has at most one solution.

5. (15%) Suppose we have an expression which is used to solve for unknown ϕ, that is

$$a_i\phi_i = b_i\phi_{i+1} + c_i\phi_{i-1} + d_i \quad 1 \le i \le N \qquad (A)$$

where a_i, b_i, and c_i, d_i are constants.

a) Write the above expression in a matrix form, $A\underline{\phi} = d$.

b) Matrix A is called a tridiagonal matrix. How do you solve for the solution? Write down the recursive expressions you may use.
(i.e. Write the P_i, Q_i in recursive forms)

Hint: ϕ_i can be first written in the form

$\phi_i = P_i\phi_{i+1} + Q_i$ (B)

and

$\phi_{i-1} = P_{i-1}\phi_i + Q_{i-1}$ (C)

Substitute (C) into (A) and compare with (B). Then P_i can be written in terms of a_i, b_i, c_i and P_{i-1}. Q_i can be written in terms of a_i, b_i, c_i, d_i, P_{i-1}, Q_{i-1}.

c) Let $c_1 = 0$, $b_N = 0$. What is P_1 and Q_1? What is the solution of ϕ_i?

6. (14%) Consider the following simultaneous equations

$$\begin{cases} 3x_1(t) + \ddot{x}_2(t) = u(t) \\ \dot{x}_1(t) + 4x_1(t) - \dot{x}_2(t) = 0 \end{cases}$$

where $u(t)$ is the unit step function and all initial conditions are zero.

a) Use the method of Laplace transform to obtain $x_1(t)$.

b) Find $\lim_{t\to\infty} x_2(t)$. [You need not to solve $x_2(t)$].

7. (14%) Let us call the initial displacement problem the problem of finding $y(x, t)$ for $0 < x < \ell$ such that

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with initial conditions

$$\begin{cases} y(x, 0) = f(x) & \text{(for given } f\text{)} \\ \dfrac{\partial y}{\partial t}(x, 0) = 0 & \text{(no initial velocity)} \end{cases}$$

and boundary conditions

$$\begin{cases} y(0, t) = 0 \\ y(\ell, t) = 0 \end{cases} \qquad \text{for all} \quad t$$

Show that if f is twice differentiable, then

$$\begin{aligned} y(x, t) &= \frac{1}{2}[f(x - ct) + f(x + ct)] \\ &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi ct}{\ell}\right). \end{aligned}$$

where the b_n are the half-interval sine coefficients

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

and f is to be extended so that it is odd periodic.