1. Evaluate the following integral

(14%)

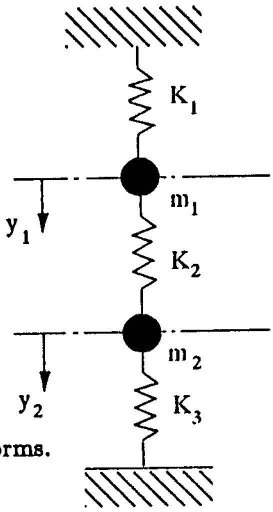
$$\int_0^\infty e^{-x^2}\cos(2ax)\,dx,$$

by the residue theorem.

Note that $\int_0^\infty e^{-x^2}\,dx = \sqrt{x}/2$.

(14%) 2. Consider the mass-spring system of Fig. 1, where y_1 & y_2 are displacements from static equilibrium positions and the positive direction is downward. Neglect the mass of spring themselves, and assume that damping is negligible. If there are external driving forces $F_1(t)$ and $F_2(t)$ acting on m_1 & m_2, respectively, then the motion is governed by

$$\begin{aligned} m_1 y_1'' &= -k_1 y_1 + k_2(y_2 - y_1) + F_1(t) \\ m_2 y_2'' &= -k_2(y_2 - y_1) - k_3 y_2 + F_2(t) \end{aligned}$$



Let

$$y_1(0) = y_1'(0) = y_2(0) = y_2'(0)$$

$$\begin{aligned} m_1 &= 4 & k_1 &= 2 \\ m_2 &= 2 & k_2 &= 2 \\ & & k_3 &= 1 \end{aligned}$$

Find $y_2(t)$ by solving the differential equations using Laplace transforms.

(14%) 3. Consider the nonlinear transformation from (x, y, z), to a new set of coordinates denoted by (q^1, q^2, q^3), and

$$\begin{aligned} x &= a - q^2\sin(\frac{q^1}{a}) \\ y &= a - (a - q^2)\cos(\frac{q^1}{a}) \\ z &= q^3 \end{aligned}$$

where (x, y, z) is the cartesian coordinate. Determine the fundamental metric g_{ij} for the new coordinate system.

(14%) 4. Solve the following integral equation for $y(x)$

$$\int_{-\infty}^{\infty}\frac{y(u)du}{(x-u)^2 + a^2} = \frac{1}{x^2 + b^2},\ 0 < a < b.$$

Hint: make use of the convolution theorem of the Fourier transformation.

Given: The Fourier transform of $\frac{1}{x^2 + c^2}$, with $c > 0$, is

$$\mathcal{F}\left\{\frac{1}{x^2 + c^2}\right\} = \int_{-\infty}^{\infty}\frac{e^{-iwx}}{x^2 + c^2}dx = \frac{\pi}{c}e^{-cw}.$$

(16%) 5. Solve the following ODEs for $y(x)$ and $P(x)$

$$\begin{cases} \frac{dy}{dx} + \frac{dP}{x} &= 2 \\ \frac{d^2y}{dx^2} - P &= \sin x \end{cases}$$

$$\text{B.C.: } \left.\frac{dy}{dx}\right|_{x=0} = y(0) = P(0) = 0$$

(14%) 6. Write down the characteristics of the following PDE and solve for $\phi(x,t)$:

$$\frac{\partial^2 \phi}{\partial t^2} = \beta \frac{\partial^2 \phi}{\partial x^2} \ , \quad \beta = \text{constant} > 0$$

$$\text{I.C.: } \phi(x,0) = x \ , \quad \left.\frac{\partial \phi}{\partial t}\right|_{t=0} = \cos x$$

(14%) 7. Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1,1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-c}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Find the sufficient and necessary conditions such that

a) f is continuous (3%)

b) $f'(0)$ exists (3%)

c) f' is continuous (4%)

d) $f''(0)$ exits (4%)