1. $f(t) = f_0(t) + f(t-b)$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$

(14%) $f_0(t) = 1$ for $0 \leq t \leq a$ and $f_0(t) = 0$ elsewhere

$a < b$, derive the Laplace transform for the function $f(t)$

2. 假設 U 為流體之速度向量，則下列之物理意義為何？

(14%)

1) div U = 0

2) curl U = 0

3) div (curl U)

並証明上述 3) 式為何。

3. a. If z is a complex variable.

(14%) i). Does $e^{\ln z}$ always equal z? Explain why.

ii). Does $\ln e^{z}$ always equal z? Explain why.

b. Evaluate the integral

$$\oint_C (z - z_o)^{-1}\, dz,$$

where the contour C encircles the point z_o in a positive (counterclockwise) sense.

4. (a) Consider the initial value problem

(15%)

$$\dot{X} = AX, \quad A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (*)$$

where $\dot{X} \equiv dX/dt$. Find the eigenvalues of A (2%) and construct the solution of the O.D.E. (*). (3%)

(b) Solve the following initial value problem (10%)

$$\dot{X} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

5. Specify methods to calculate a) the row space, b) the column space and c) the null space of a matrix A. Use the specified method to show that the vector $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 6\mathbf{k}$ where $\mathbf{i}$, $\mathbf{i}$ and $\mathbf{k}$ are unit vectors along the cartesian coordinate axis can not be expressed in terms of linear combination of the vectors $\mathbf{u}_1 = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{u}_2 = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{u}_3 = 2\mathbf{i} + \mathbf{k}$.

(14%)

6. Solve the problem

(14%)

$$u_t = u_{xx} + \sin(3\pi x) \qquad 0 < x < 1$$

with BCs

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

and IC

$$u(x,0) = \sin(\pi x) \qquad 0 \leq x \leq 1$$

(15%) 7. Please find the derivatives for the following functions.

(a) $f(x) = \sqrt{1 + \sqrt{2 + \sqrt{4 + x^3}}}$

(b) $[A(x)]^{-1} = \begin{bmatrix} 1+x & 1 & 2 \\ 2x & \log x & 1 \\ x^2 & x & 1/x \end{bmatrix}^{-1}$

(c) $I(x) = \int_x^{x^2} \log(x+y)dy.$

Note that the final solution for problem (b) may be expressed as

$$\frac{d([A(x)]^{-1})}{dx} = [A(x)]^{-1}[B(x)][A(x)]^{-1}.$$

It is unnecessary to implement the inversion and multiplication. Just write down the solution for the matrix $[B(x)]$.