1. **(14 points)**

Evaluate the following integral

$$\int_0^\infty \frac{dx}{x^2+1}$$

by using the *Residue Theorem* in the complex variables theory.

2. **(14 points)**

Assume $\{\phi_n(x)\}$, $n = 0, 1, 2, \ldots$, is a normalized orthogonal set which is applied to expand a given function $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x)$$

where A_n is known and $\alpha \le x \le \beta$. Now we want to approximate $f(x)$ by $f_m(x)$ defined as

$$f_m(x) = \sum_{i=0}^{m} r_i \phi_i(x)$$

The problem is to find r_i, $i = 0, 1, \ldots, m$, such that

$$J = \int_\alpha^\beta [f(x) - f_m(x)]^2 \, dx$$

is minimized. Express your answer r_i in terms of A_i.

3. **(14 points)**

Find the solution of the following set of equations:

$$2\frac{d^2x}{dt^2} - \frac{dy}{dt} - 4x = 2t$$

$$2\frac{dx}{dt} + 4\frac{dy}{dt} - 3y = 0$$

4. **(15 points)**

Given a set of parametric equations

$$x = t\cos t$$

$$y = 2t\sin t$$

$$z = t^2$$

Determine the unit tangent, principal normal, binomal vectors, and the radii of curvature and torsion as functions of t. Also, determine the osculating, normal, and rectifying planes at $t = 2$.

5. **(14 points)**

a). If $f(x_1, x_2, ..., x_n) = x_1^k + x_2^k + ... + x_n^k$, show that a local extremum of f, subject to the condition of $x_1 + x_2 + ... + x_n = a$, is $a^k n^{1-k}$.

b). Find the maximum and minimum values of $2x^2 + y^2 + 2x$ for $x^2 + y^2 \leq 1$.

6. **(15 points)**

a). Produce a matrix that diagonalizes the given matrix, or show that this matrix is not diagonalizable:

a). $\begin{pmatrix} 0 & -1 \\ 4 & 3 \end{pmatrix}$,

b). $\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$,

c). $\begin{pmatrix} 5 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & -2 \end{pmatrix}$.

7. **(14 points)**

Consider the partial differential equation of

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a}\frac{\partial u}{\partial t}$$

with the boundary conditions of

$$u = b \quad \text{at } x = 0, t > 0$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{at } x = 1, t > 0,$$

and the initial condition of

$$u = 0 \quad \text{for } t = 0, 0 \leq x \leq 1$$

Obtain the analytical solution as a function of x and t.