1. Solve the following initial or boundary value problems.

(a) $y' = [y - 4x]^2$, $y(0) = 0$ (12%)

(b) $y'' - 2y' + y = e^x$, $y(1) = 1$, $y(-1) = 1$ (13%)

2. (15%)

Using the Laplace transform to solve for $y(t)$ from the simultaneous equations

$$\frac{dy}{dt} + 2y + 6\int_0^t z\, dt = -2u(t),$$

$$\frac{dy}{dt} + \frac{dz}{zt} + z = 0.$$

Where $y(0) = -5$, $z(0) = 6$ and $u(t)$ denotes the unit step function.

3. (15%)

Find the eigenvalues and eigenvectors of the matrix $[A] = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

(Hint: eigenvalues of $[A]$ are integers.)

4. (15%)

Let $x = 2s + t^3$ and $y = 2t + s^2$,

a) Compute $\frac{\partial x}{\partial s}$ and $\frac{\partial y}{\partial t}$.

b) Compute $\frac{\partial s}{\partial x}$ and $\frac{\partial t}{\partial x}$.

c) Let $u = s^2 + t^3$, compute $\frac{\partial u}{\partial x}$.

5. (15%) Apply the calculus of residues to find

$$\int_0^{2\pi} \frac{d\theta}{1+\varepsilon\cos\theta}$$

where $|\varepsilon|<1$.

6. (15%) Assume $\phi(x)$ has the Fourier sine expansion of form:

$$\phi(x)=\sum_{n=1}^{\infty} a_n \sin(n\pi x) \qquad (1)$$

What is the difference between Eq. (1) and the solution of the following partial differential equation?

PDE $\quad u_t=\alpha^2 u_{xx} \quad 0<x<1, \quad 0<t<\infty$

BCs $\quad \begin{cases} u(0,t)=0 \\ u(1,t)=0 \end{cases} \quad 0<t<\infty$

IC $\quad u(x,0)=\phi(x), \quad 0\le x\le 1$