1. a) (7%) Prove the Cauchy principle value theorem.

   b) (7%) Evalute the real integral

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx\ ; \qquad a > 0,\ \text{real}\ .$$

2. (6%) Let $F(z) = \sqrt{z^2 - 1}$, where z is complex variable.
Evalute $F(x+0i)$ and $F(x-0i)$, where $-1 < x < 1$ and $i = \sqrt{-1}$.

3. Solve the first order partial differential equation with the initial condition

$$u(x,0) = \sin x, \text{ for } -\infty < x < \infty .$$

$$\frac{\partial\, u(x,t)}{\partial\, x} + \frac{\partial\, u(x,t)}{\partial\, t} + u(x,t) = 0;\ -\infty < x < \infty,\ 0 < t < \infty,$$

by

a) (10%) Method of Characteristics ;
b) (10%) Method of Laplace Transform ;
c) (10%) Method of Fourier Transform .

Hint:

$$\mathcal{L}(\sin wt) = \frac{w}{s^2 + w^2} \quad ; \quad \mathcal{L}(\cos wt) = \frac{s}{s^2 + w^2} .$$

4. (12%) Solve

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$$

5. (12%) Solve $x^2y'' + x(2-x)y' - 2y = 0$.

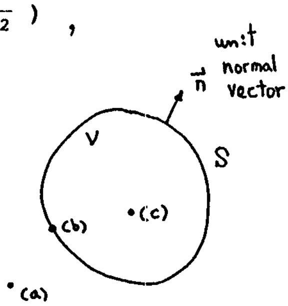
6. (14%) Find eigenvalue, eigenfunction and orthogonality relationship of

$$\begin{cases} x^2\, y'' + x\, y' + \lambda^2\, y = 0, & \lambda > 0; \\ y(1) = 0 \text{ and } y(e^{\pi}) = 0 . \end{cases}$$

7. (12%) In a fluid domain, V is bounded by a smooth closed surface, S and

$$\begin{cases} \nabla^2\phi = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \right) \phi = 0 \ ; \\ \nabla^2\psi = \nabla^2 \left(\dfrac{1}{4\pi\, [(x_1-x)^2 + (y_1-y)^2 + (z_1-z)^2]^{1/2}} \right) , \\ \qquad = \delta\,(\vec{x}_1 - \vec{x}\,)\ ; \end{cases}$$

where $\vec{x}_1 = (x_1, y_1, z_1)$; $\vec{x} = (x, y, z)$.



Calcuate

$$\iint_{S(x,y,z)} \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n} \right) dS$$

when $\vec{x}_1$ is (a) Outside S; (b) On S; and (c) Inside S.