1. Solve the initial value problem. (10%)

$$y''' + 3y'' + 3y' + y = 30e^{-x},\ y(0) = 3,\ y'(0) = -3,\ y''(0) = -47$$

2. Solve the differential equation by **power series method**. (10%)

$$y'' + y = 0$$

3. Solve the differential equation by **Laplace Transform**. (10%)

$$y'' + y = 2t,\ y\left(\frac{\pi}{4}\right) = \frac{\pi}{2},\ y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$$

4. Explain the Fourier Transform in detail and describe the physical meaning of Fourier Transform. (10%)

5. Find the inverse of matrix **A**, if it exists. (10 %)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

6. (a) Find an LU factorization of matrix **A**, where L is a lower triangular matrix and U is upper triangular matrix. (7%)

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

(b) Explain the purposes of LU factorization. (3%)

7. Let matrix **A** be a $n \times n$ invertible matrix. Check that if the following statements are equivalent. Answer true or false. (10%)

(a) The equation **A**x=b is consistent and has infinite solutions.

(b) The columns of **A** form a linearly dependent set.

(背面仍有題目,請繼續作答)

(c) **A** has n pivot positions.

(d) $\mathbf{A}^T$ is an invertible matrix.

(e) The linear transformation $x \mapsto \mathbf{A}x$ is one-to-one and onto.

(f) det**A**=0.

(g) rank**A**=n.

(h) dim(Col**A**)=dim(Nul**A**)=n.

(i) The columns of **A** form a basis of R^n.

(j) **Ax**=0 has only the trivial solution.

8. (a) Find a least-squares solution of **Ax=b** for (7%)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

(b) Explain the geometric meaning of least-squares solutions (3 %)

9. Diagonalize the matrices if possible ($A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D). (10%)

(a) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

10. Given a scalar function $\mathbf{u}(x, y, z) = zy + yx$ and a vector function $\mathbf{v}(x, y, z) = [y, z, 4z - x]$, find (a) $\nabla\mathbf{u}$, (b) $\nabla \cdot \mathbf{v}$, (c) $\nabla \times \mathbf{v}$, (d) $\nabla^2\mathbf{u}$, (e) $\nabla \times (\nabla \times \mathbf{v})$, where ∇ is the gradient operator and ∇^2 is the Laplace operator (10%)