※ 考生請注意：本試題不可使用計算機

1. Given a nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = r(x),\ r(x) \neq 0 \qquad (1)$$

and its corresponding homogenous equation

$$y'' + p(x)y' + q(x)y = 0 \qquad (2)$$

Prove that the sum of a solution of Eq. (1) and a solution of Eq. (2) is a solution of Eq. (1). (10 %)

2. Let $f(x)$ be a periodic function of period 2π. Prove that $f(x)$ can be represented by the trigonometric series: (15%)

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \cdots$$

3. Find the Fourier series of the function (10%)

$$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ k, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}$$

4. Show that the vector set **V** is linearly independent. (10 %)

$$\mathbf{V} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \right\}$$

5. Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$.

(a) Show that T is a one-to-one linear transformation. (5 %)

(b) Does T map $\boldsymbol{R}^2$ onto $\boldsymbol{R}^3$? (show the details) (5 %)

※ 考生請注意：本試題不可使用計算機

6. Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the bases for $\boldsymbol{R}^2$ given by $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2]$ and $\mathbf{C} = [\mathbf{c}_1, \mathbf{c}_2]$. Find the change-of-coordinates matrix from **B** to **C**. (10 %)

7. (a) Diagonalize matrix **A**, that is, find an invertible matrix **P** and a diagonal matrix **D** such that $\mathbf{A=PDP^{-1}}$. (15 %)

$$\mathbf{A} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

(b) Explain the purpose of matrix diagonalization. (5 %)

8. Find a least-squares solution of the inconsistent system $\mathbf{A}x = \mathbf{b}$ for (15 %)

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$$