※ 考生請注意：本試題不可使用計算機。請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. Given a homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Support that we know a solution y_1, which is not identically zero. Find another solution y_2 by using the method of order reduction, such that $\{y_1, y_2\}$ forms a solution basis (10 %).

2. The Fourier series of a function $f(x)$ with period $2L$ is given by

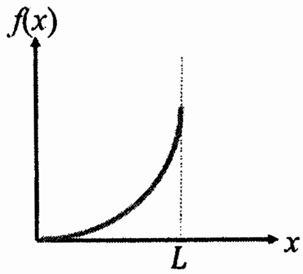
$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, \cdots$$



Expand the function $f(x) = x^2, 0 < x < L$ in (a) a cosine series, and (b) a Fourier series (containing cosine and sine terms) (20%).

3. Find the inverse of matrices if possible (10%).

(a) $\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{bmatrix}$

4. Let $\mathbf{A}$ be a $n \times n$ symmetric matrix. Prove that the eigenvectors of $\mathbf{A}$ corresponding to different eigenvalues are orthogonal (10%).

※ 考生請注意：本試題不可使用計算機。請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

5. Given a symmetric matrix **A**, find an orthogonal matrix **S** such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is diagonal (15%).

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

6. (a) Drive that $\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$ is the optimal solution of the least-square system $\mathbf{Ax} = \mathbf{b}$

(b) Find a least-square solution of $\mathbf{Ax{=}b}$ for (15%)

$$\mathbf{A} = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

7. Let **A** be a $m \times n$ matrix with rank r. The singular value decomposition (SVD) decomposes matrix **A** into $\mathbf{U\Sigma\Sigma}^T$, where **U** is an $m \times m$ orthogonal matrix, **V** is an $n \times n$ orthogonal matrix, and $\mathbf{\Sigma}$ contains r singular values of **A** in the diagonal entries. Find a singular value decomposition of matrix **A** (20%).

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$