※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. If both square matrices B and C are inverses of a square matrix A, then B = C. In other word, if A has an inverse, the inverse is unique. Please prove the uniqueness. (10%)

2. Please prove the following addition formulas for sine and cosine: (15%)

$$\begin{cases} \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \\ \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \end{cases}$$

(***Hint***: You can use the linear transformation y=Ax with a rotation matrix A to derive it.)

3. Given a real square matrix A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, this matrix A is orthogonal.

   (1) Please explain the definition of an orthogonal matrix and prove that A is an orthogonal matrix. (10%)

   (2) An orthogonal transformation preserves the value of the inner product of vectors. What is an orthogonal transformation? Please verify that it holds for this matrix A. (15%)

4. Let *R* be a closed bounded region in the *xy*-plane whose boundary *C* consists of finitely many smooth curves. After derivation, we have *A* = $\frac{1}{2}\oint_C(xdy-ydx)$. This formula expresses the area of *R* in terms of a line integral over the boundary *C*. The theory of some **planimeters(求積儀)** is based on it. Please derive this formula *A* = $\frac{1}{2}\oint_C(xdy-ydx)$. (20%)

5. In a space defined by the right-handed Cartesian coordinate system, let $\vec{i}, \vec{j}, \vec{k}$ be the three unit vectors in the directions of positive *x*-, *y*- and *z*-coordinate axes, respectively. Let a particle A of mass *M* be fixed at a point $P_0(x_0, y_0, z_0)$ and let a particle B of mass *m* be free to take up various positions $P(x, y, z)$ in space. Then A attracts B. According to **Newton's law of gravitation** the corresponding gravitational force $\vec{p}$ is directed from *P* to P_0, and its magnitude is proportional to $1/r^2$, where r is the distance between *P* and P_0, say,

$$|\vec{p}| = c / r^2, \quad \text{with } c = GMm$$

where G ($=6.67 \cdot 10^{-8}\,\mathrm{cm}^3\,\mathrm{g}^{-1}\,\mathrm{sec}^{-2}$) is the gravitational constant. Hence $\vec{p}$ defines a vector field in space.

(1) Please prove the gravitational force vector $\vec{p}=$

$$\frac{-c(x-x_0)}{r^3}\cdot\vec{i}+\frac{-c(y-y_0)}{r^3}\cdot\vec{j}+\frac{-c(z-z_0)}{r^3}\cdot\vec{k}. \text{ (10\%)}$$

(2) The gradient of a given scalar function $f(x, y, z)$ is the vector function defined by

grad $f = \frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}$. Please show that the force $\vec{p}$ is the gradient of the scalar function $f(x, y, z) = c/r$, which is a potential of that gravitational field. (10%)

(3) Please prove that f satisfies the following famous **Laplace's equation**: (10%)

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0.$$