編號：G 223 系所：測量及空間資訊學系 科目：工程數學

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1. Suppose that the linear relationship between the observation vector ℓ and the parameter vectors x and y is expressed as follows:

$$\ell + v = Ax + By, \qquad \Sigma_{\ell\ell} = \sigma_0^2 P^{-1}$$

If the prior information of the parameter vector y is assumed to be $(y_0, \Sigma_{yy}) = (y_0, \sigma_0^2 P_y^{-1})$ and y and ℓ are uncorrelated, Using the least-squares principle, find the estimation for the parameter vectors x and y, and show the estimation is the unbiased. In the formulae, A and B are coefficient matrices; $\Sigma_{\ell\ell}$ and Σ_{yy} are the covariance matrices; P and P_y are the weight matrices, and σ_0^2 is the variance factor of unit weight. (20 points)

2. The manufacturer's specified standard error for an EDM instrument is 5 mm + 10 ppm. Assume that other errors in EDM are neglected and the measurements are assumed to be the normal random variables.
   (1) Calculate the standard deviation in a distance measurement of 10 km. (5 points)
   (2) If the distance in (1) is measured four times, what is the standard deviation in the average distance? (5 points)
   (3) Find the 95% confidence interval for the average distance in (2). (5 points)
   (4) The distance in (1) is now re-measured four times by another EDM instrument with the specified standard error 3 mm + 3 ppm. Find the maximum difference limit of these two average distances at a 5% significant level, if these two average distances are not significantly different. (5 points)

Note that the values of the standard normal deviate is 1.65 for a 95% probable error and 1.96 for a 97.5 % probable error.

standard normal deviate = $\dfrac{x-\mu}{\sigma_x}$,

x: normal distributed random variable; μ: the expectation of x;
σ_x : the standard error of x

(背面仍有題目,請繼續作答)

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3. In a space defined by the right-handed Cartesian coordinate system, let $\vec{i}, \vec{j}, \vec{k}$ be the three unit vectors in the directions of positive x-, y- and z-coordinate axes, respectively. Let a particle A of mass M be fixed at a point $P_0(x_0, y_0, z_0)$ and let a particle B of mass m be free to take up various positions $P(x, y, z)$ in space. Then A attracts B. According to **Newton's law of gravitation** the corresponding gravitational force $\vec{p}$ is directed from P to P_0, and its magnitude is proportional to $1/r^2$, where r is the distance between P and P_0, say,

$$|\vec{p}| = c/r^2, \quad \text{with } c = GMm$$

where G $(=6.67 \cdot 10^{-8}\ \text{cm}^3\,\text{g}^{-1}\,\text{sec}^{-2})$ is the gravitational constant. Hence $\vec{p}$ defines a **vector field** in space.

(3a) Please derive the function of the gravitational force vector $\vec{p}$. (5 points)

(3b) The **gradient** of a given scalar function $f(x, y, z)$ is the vector function defined by

$$\text{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

Please show that the force $\vec{p}$ is the gradient of the scalar function $f(x, y, z) = c/r$, which is a **potential** of that **gravitational field**. (5 points)

(3c) Please verify that f satisfies the following famous **Laplace's equation**:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0. \text{ (5 points)}$$

(3d) The curl of the gravitational force vector $\vec{p}$ is defined by

$$\text{curl } \vec{p} = \nabla \times \vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p_1 & p_2 & p_3 \end{vmatrix}, \text{ with } \vec{p} = p_1\vec{i} + p_2\vec{j} + p_3\vec{k}.$$

Please find the curl $\vec{p}$. (5 points)

4. Suppose that a continuous function $f(x)$ is discretized into a sequence

$$\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots, f(x_0+[N-1]\Delta x)\}$$

by taking N samples Δx units apart. Here, one uses x as a discrete variable and defines

$$f(x) = f(x_0 + x\Delta x)$$

where x now assumes the discrete values 0, 1, 2, …, N-1. In other words, the sequence $\{f(0), f(1), f(2), \ldots, f(N-1)\}$ denotes any N uniformly spaced samples from a corresponding continuous function. Herewith, one defines the **discrete Fourier transform (DFT)** as follows:

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$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)e^{-j2\pi ux/N} = \frac{1}{N}\sum_{x=0}^{N-1} f(x)W_N^{ux} \qquad (1)$$

with $W_N = e^{-j2\pi/N}$ and $j = \sqrt{-1}$. Let $N = 2M = 2^n$, one can get

(4a) $W_{2M}^{2ux} = W_M^{ux}$, (4b) $W_M^{u+M} = W_M^u$, and (4c) $W_{2M}^{u+M} = -W_{2M}^u$. Please prove them. $(3 \times 2 = 6$ points)

(4d) Let N=2M, Eq. (1) gives

$$F(u) = \frac{1}{2}\left[F_{even}(u) + F_{odd}(u)W_{2M}^u\right] \qquad (2)$$

with $F_{even}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x)W_M^{ux}$ and $F_{odd}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)W_M^{ux}$

for u = 0 (1) M-1. Please prove it. (5 points)

(4e) Hence, one can get $F_{even}(u+M) = F_{even}(u)$ and $F_{odd}(u+M) = F_{odd}(u)$. Please prove it. $(2 \times 2 = 4$ points)

(4f) Hence, Eq. (2) gives $F(u+M) = \frac{1}{2}\left[F_{even}(u) - F_{odd}(u)W_{2M}^u\right]$. Please prove it. (5 points)

5. The earth can be approximated by a sphere S of radius R. This sphere S also can be represented by $\vec{r}(u,v)$ = [$R\cos v\cos u$, $R\cos v\sin u$, $R\sin v$], where $\vec{r}$ denotes the position vector of any point P on S and $0 \le u \le 2\pi$, $-\pi/2 \le v \le \pi/2$.

(5a) Please compute the partial derivative vectors $\vec{r}_u$ and $\vec{r}_v$. $(2 \times 2 = 4$ points)

(5b) Please determine the normal vector of S at P by the vector product $\vec{r}_u \times \vec{r}_v$. (3 points)

(5c) Please determine the vector length $|\vec{r}_u \times \vec{r}_v|$. (2 points)

(5d) The area of such a sphere S can be determined by the ***surface integral*** A(S) = $\iint |\vec{r}_u \times \vec{r}_v|\, du\, dv$. Please prove that such a sphere has the area A(S) = $4\pi R^2$ by the surface integral A(S). (6 points)

(5e) Please determine the unit normal vector of S at any point $P(x, y, z)$ on S. (5 points)

Note in (4d) that " u = 0 (1) M-1" denotes "u = 0, 1, 2, 3, ..., M-1".