1. (A) Find a solution of the equation

$$(1 - x^2)\,dy + 4xy\,dx = 0$$

with the property that $y=9$ when $x=-2$, $y=2$ when $x=0$, and $y=0$ when $x=2$. 10%

(B) Solve $y^{(4)} - 2y''' + 2y'' - 2y' + y = x\sin x$. 10%

2. Solve the following equations:

(A) $\begin{cases} \dfrac{d^2y}{dx^2} = z + x , \\ \dfrac{d^2z}{dx^2} = y + 2x ; \end{cases}$

(B) $\begin{cases} y' + 2y + 6\displaystyle\int_0^t z(\tau)\,d\tau = -2u(t), \\ y' + z' + z = 0. \end{cases}$

$y(0) = -5$, $z(0) = 6$, and $u(t)$ is the unit step function: $u(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t > 0. \end{cases}$ 20%

3. For what nonzero values of k, if any, does the equation $y^{(4)} - k^4 y = 0$ have nontrivial solutions(that is , $y \neq 0$) which satisfy the conditions

$$y(0) = y''(0) = y(b) = y''(b) = 0 ?$$

Where b is a positive number. What are these solutions if they exist ? 20%

4. (A) Using the series method to solve $2x^2y'' + (x^2 - x)y' + y = 0$. 10%

(B) If y_1 and y_2 are linearly independent solutions of the equation

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0$$

over an interval I on which $a_0(x) \neq 0$, prove that between any two consecutive zeros of either of these solutions in I there is exactly one zero of the other. 10%

5. Two masses, $m_1 = 1$ and $m_2 = 2$, are connected by springs of moduli $k_1 = 1$, $k_{12} = 2$, and $k_2 = 2$, as shown in the following figure. Neglecting all frictional effects and assuming that each spring is unstretched when the system is in its equilibrium position, determine the frequencies of the free vibrations of the system and discuss the motion of the system at each of these frequencies. 20%