**** You must solve the problems in consecutive order. ****

1. A solid slab, which occupies $0 < X < \ell$, is initially at a constant temperature, U_0. Both ends are held at a constant temperature of zero. Show that the temperature at any position in the slab is given by: (15 points)

$$U(x,t) = U_0 \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right) + U_0 \sum_{n=1}^{\infty} (-1)^{n+1} \left\{ \operatorname{erf}\left(\frac{n\ell - x}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{n\ell + x}{2\sqrt{kt}}\right) \right\}$$

where k is the thermal diffusivity.

Note: The governing equation is:

$$\frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2}$$

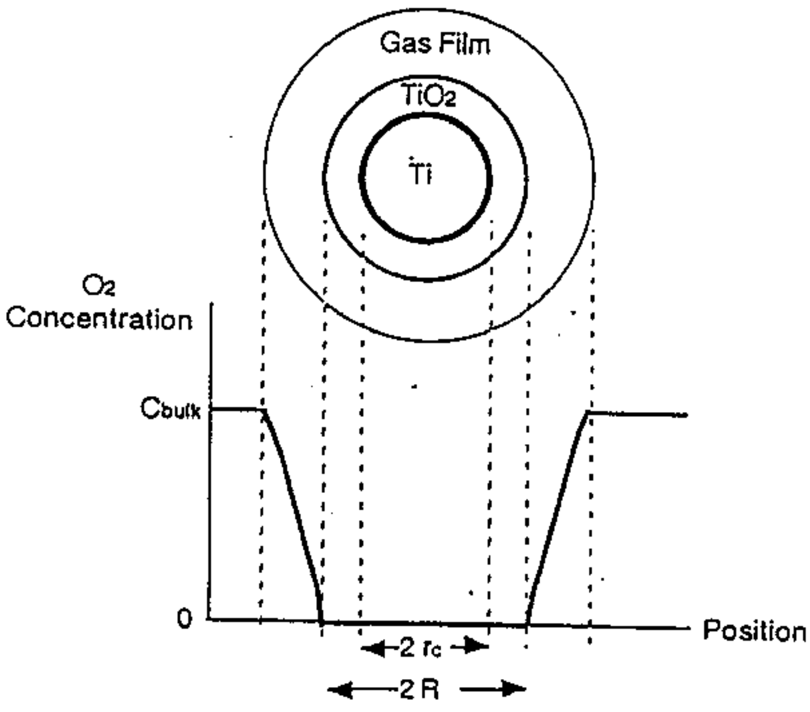
2. Spherical particles of solid titanium at elevated temperature are contacted with a preheated stream of air. Oxidation occurs according to the chemical reaction:

$Ti\,(s) + O_2\,(g) \longrightarrow TiO_2\,(s)$

which is irreversible. An adherent TiO_2 scale forms on the Ti spheres.. The concentration profile of O_2 across the gas film around the particle, TiO_2, and Ti particle is shown below. Assume the control mechanism for the oxidation is oxygen transport through the gas film to the surface of the TiO_2, and the diffusion flux through the gas film at TiO_2 surface is J (flux) = $-k_g\,\Delta C$, where k_g is the mass transfer coefficient (cm/s), and ΔC is the concentration gradient across the gas film (mole/cm³). Show the oxidation time is given by: (25 points)

$$t = \frac{\rho R}{3 k_g C_{bulk}} \left[1 - \left(\frac{r_c}{R}\right)^3 \right]$$

where ρ is density of Ti.



3. An important task of engineering mathematics is to model an engineering or physical phenomenon with mathematical equation. Many phenomena can be interpreted by some linear equations of the form

$$\nabla^2 U = a \frac{\partial^2 U}{\partial t^2} + b \frac{\partial U}{\partial t},$$

where U is a function of both position and time (t), a and b are some physical constants, ∇^2 is the Laplacian operator in one-, two- or three-dimensional space. The governing equation of Problem 1 (i.e., the equation of heat flow with U as the temperature) is an example where a is zero. Diffusion equation (Fick's second law) with constant diffusion coefficient is another example where a is also zero. On the other hand, b=0 leads to various wave equations. Choose from any of the above examples or others to derive an equation which involves the Laplacian operation, ∇^2, and time derivative of some function. (20 points)

4. Many materials have laminar crystal structure and thus possess anisotropic electrical behavior in which the electrical conductivity along the laminar plane is the greatest. Assuming Ohm's law (i.e., current density is linearly proportional to electric field), the current density is parallel to the electric field if the electric field is applied along a direction in the laminar plane or perpendicular to the laminar plane. Show that for a general two-dimensional Cartesian coordinate (x, y) which is inclined to the laminar plane (i.e., both x-axis and y-axis are not on the laminar plane), the current density along x-direction is a linear function of both the electric field along x-direction and that along y-direction. (20 points)

5. Solve the following initial value problem: (20 points)

$$\begin{cases} \dfrac{dy_1(t)}{dt} = 2y_1(t) - 4y_2(t) \\ \dfrac{dy_2(t)}{dt} = y_1(t) - 3y_2(t), \end{cases}$$

$$\begin{cases} y_1(0) = 5 \\ y_2(0) = 2. \end{cases}$$