1. Find the following limits:

   (i) $\lim\limits_{u\to 0^+} \frac{\int_0^{\sin u} \sqrt{\tan x}dx}{\int_0^{\tan u} \sqrt{\sin x}dx}$ provided that $\lim\limits_{x\to 0^+} \frac{\sin x}{\tan x} = 1$ is known. 5%

   (ii) $\lim\limits_{x\to 0} \frac{e^x+e^{-x}-2}{1-\cos x}$. 5%

   (iii) $\lim\limits_{x\to\infty} x\sin\frac{1}{x}$. 5%

   (iv) $\lim\limits_{x\to 1} \frac{x+\sqrt{x}-2}{x^3-1}$. 5%

2. Find the following integrals:

   (i) $\int \sin 3x \cos^5 x dx$. 5%

   (ii) $\int\int_\Omega (x+3y^3)\, dxdy$ where $\Omega = \{(x,y)\in \mathbb{R}^2 \mid 0 \le x^2+y^2 \le 1\}$. 5%

   (iii) $\int_0^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$. 5%

   (iv) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(x-y)^2}}{1+(x+y)^2} dxdy$. 5%

3. If f is a continuous function in $[0,1]$ and $f(x) < 1$ for $x \in [0,1]$. Then equation $2x - \int_0^x f(t)dt = 1$ has one and only one root in $[0,1]$. 10%

4. Show that the function $f(x) = e^x + x$ being differential and one-to-one has a differentiable inverse $f^{-1}(x)$ and find the value of $\frac{df^{-1}}{dx}$ at point $f(\ln 2)$. 10%

5. If a is a positive constant, show that

$$\int_0^\infty e^{-a^2x^2}dx = \frac{1}{a}\int_0^\infty e^{-x^2}dx.$$

   Use this fact to show that the centroid of the region between the curve $y = e^{-a^2x^2}$ and the x-axis is $(0, \frac{\sqrt{2}}{4})$. 10%

6. Use the alternating series test to show that the improper integral

$$\int_0^\infty \frac{\sin x}{x}dx$$

   converges. 10%

7. Find the real number C such that

$$\int_{-\infty}^{C} xe^{2x}dx = \lim_{x\to\infty}\left(\frac{x+C}{x-C}\right)^x.$$

   10%

8. Show that $\sum\limits_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln\ln n}}$ diverges. Hint: show that $(\ln\ln n)^2 \le \ln n$ for large n. 10%

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