國立成功大學

115學年度碩士班招生考試試題

編　號：127

系　所：電機工程學系

科　目：離散數學

日　期：0203

節　次：第 3 節

注　意：1.不可使用計算機
2.請於答案卷(卡)作答，於試題上作答，不予計分。

1. (10 points) An universal operator is **nor**, defined such that
   P nor Q ⇔ ¬(P ∨ Q).
   Show how to express **P ∧ Q** in terms of: **nor, P, Q,** and grouping parentheses.

2. (10 points)We define the sequence of numbers

$$a_n = \begin{cases} 1 & \text{if } 0 \le n \le 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \ge 4. \end{cases}$$

   Prove that $a_n \equiv 1 \pmod 3$ for all $n \ge 0$.

3. (20 points)An outerplanar graph is an undirected graph for which the vertices can be placed on a circle in such a way that no edges (drawn as straight lines) cross each other. For example, the complete graph on 4 vertices, K_4, is not outerplanar but any proper subgraph of K_4 with strictly fewer edges is outerplanar.

   Prove that any outerplanar graph is **3-colorable.** A fact you may use without proof is that any outerplanar graph has a vertex of degree at most **2**.

4. (20 points)Let n be any integer of the form m^2 for some positive integer m. Let S_n be the set of all pairs (i,j) such that (i) i,j are non-negative integers, and (ii) $i^2 + j \le n$. Let $h(n) = |S_n|$. Determine $h(n)$.

5. (a) (10 points)Let

$$f(n) = \sum_{0 \le k \le n} (-1)^k k.$$

   Prove that

$$f(n) = (-1)^n \left\lceil \frac{n}{2} \right\rceil.$$

   (b) (10 points)Let $g(n) = \sum_{k \ge 0} \frac{1}{4^k} \binom{n+k-1}{k}$. Determine $g(n)$.

6. (20 points)Let $T_1, \ldots, T_{100}$ be **mutually independent** random variables, where each T_i represents the fraction of a day a student needs to solve the i-th homework problem. You only know that for all i,

$$0 \le T_i \le 1, \mathbb{E}[T_i] = 0.3.$$

   Let $T = \sum_{i=1}^{100} T_i$. Prove that

$$\Pr\{T > 30e\}$$

   is exceedingly small, and derive the **best bound you can** from the given information. (Here e is Euler's constant.)