(10%)
1. Solve the following recurrence relations by generating functions.

a) $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$, $a_0 = 10$, $a_1 = 41$

b) $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$, $a_0 = 6$, $a_1 = 20$, $a_2 = 60$.

(10%)
2. In how many ways can 30 books be distributed among 3 people A, B, and C so that

a) A and B together receive exactly twice as many books as C?

b) C receives at least two books, B receives at least twice as many books as C, and A receives at least 3 times as many books as B?

(15%)
3. a) Let $G = (V, E)$ be a connected bipartite undirected graph with V partitioned as $V_1 \cup V_2$. Prove that if $|V_1| \neq |V_2|$, then G cannot have a Hamilton cycle.

b) Prove that if the graph G in part (a) has a Hamilton path, then $|V_1| - |V_2| = \pm 1$.

c) Give an example of a connected bipartite undirected graph $G = (V, E)$, where V is partitioned as $V_1 \cup V_2$ and $|V_1| = |V_2| - 1$, but G has no Hamilton path.

(10%)
4. If $A = \{1, 2, 3, 4, 5, 6, 7\}$ define R on A by $(x, y) \in R$ if $x - y$ is a multiple of 3.

a) show that R is an equivalence relation on A.

b) Determine the equivalence classes and partition of A induced by R.

(15%)
5. Let $G = (V, E)$ be an undirected graph with subset I of V an independent set. For each $a \in I$ and any Hamilton cycle C for G, there will be $\deg(a) - 2$ edges in E that are incident with a and not in C. Therefore there are at least $\sum_{a \in I} [\deg(a) - 2] = \sum_{a \in I} \deg(a) - 2|I|$ edges in E that do not appear in C.

a) Why are these $\sum_{a \in I} \deg(a) - 2|I|$ edges distinct?

b) Let $v = |V|$, $e = |E|$. Prove that if $e - \sum_{a \in I} \deg(a) + 2|I| < v$, then G has no Hamilton cycle.

c) Select a suitable independent set I and use part (b) to show that the graph in Fig 1 has no Hamilton cycle.

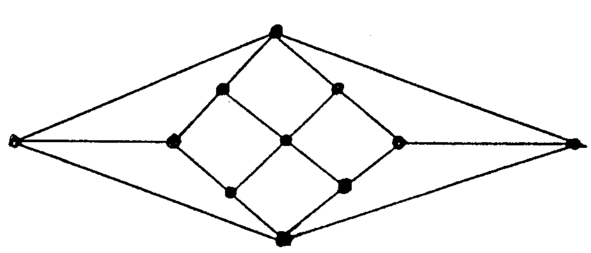

Fig 1 :

(15%)

6. a) Let $G=(V,E)$ be a loop-free undirected graph with $|V|=n$. Prove that G is a tree if and only if $P(G,\lambda)=\lambda(\lambda-1)^{n-1}$. Note that $P(G,\lambda)$ represents the number of different ways we can properly color the vertices of G using at most λ colors.

b) Prove that $\chi(G)=2$ for any tree with two or more vertices, where $\chi(G)$ is the chromatic number of G.

c) If $G=(V,E)$ is a connected undirected graph with $|V|=n$, prove that for any integer $\lambda \geq 0$, $P(G,\lambda) \leq \lambda(\lambda-1)^{n-1}$.

(15%)

7. a) Let $G=(V,E)$ be the bipartite graph shown in Fig. 2, with V partitioned as $X \cup Y$. Determine the deficiency of graph G, $\delta(G)$, and a maximal matching of X into Y.

b). For any bipartite graph $G=(V,E)$, with V partitioned as $X \cup Y$, if $\beta(G)$ denotes the independence number of G, show that $|Y| = \beta(G) - \delta(G)$.

c) Determine a largest maximal independent set of vertices for the graphs shown in Figs 2 and 3.

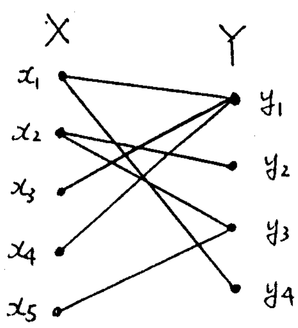

Fig. 2

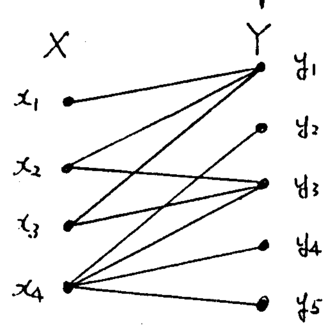

Fig. 3

8. (10%)

a). Write an expression for a_r, where a_r is the coefficient of x^r in the following generation function $A(x)$:

$$A(x) = \frac{1}{1-x} + \frac{5}{1+2x} + \frac{7}{(1-x)^5}$$

b). Find the coefficient of x^{15} in $(x^3-5x)/(1-x)^3$.