1. The probability density function (pdf) of random variable X_1 and X_2 are given below.

$$f_{X_1}(x_1) = \Pi(x_1) = \begin{cases} 1, & |x_1| \le \frac{1}{2} \\ 0, & otherwise \end{cases}$$

$$f_{X_2}(x_2) = \frac{1}{2}\Pi(\frac{x_2-1}{2}) = \begin{cases} \frac{1}{2}, & |x_2 - 1| \le 1 \\ 0, & otherwise \end{cases}$$

A random variable X is defined as $X = X_1 + X_2 + 1$

(a) Calculate the pdf of X, written symbolically as $f_X(x)$.(10%)

(b) Plot $f_X(x)$, and indicate the values relevant to the problem in the plot.(5%)

2. Let the joint probability density function (pdf) of the random variables X, Y be

$$p_{X,Y}(x,y) = x e^{-x(1+y)} u(x)u(y).$$

(a) Are X and Y statistically independent? Why.(5%)

(b) Find the probability that X>2 and Y>0.(5%)

(c) What is the pdf of y given that X=1?(10%)

3.
$$x(t) = A\Pi(\tfrac{t-1}{4}) = \begin{cases} A, & |t-1| \le 2 \\ 0, & otherwise \end{cases}$$

The Fourier transform of x(t), written symbolically as X(f), is expressed in terms of amplitude and phase as

$$X(f) = |X(f)|\, e^{j \angle H(f)}, \quad -\pi \le \angle H(f) \le \pi$$

(a) Calculate X(f).(10%)

(b) Plot |X(f)| versus f.(5%)

(c) Plot $\angle X(f)$ versus f.(5%)

In the plots in (b) and (c), you have to indicate the values relevant to the problem.

Plots not satisfying |X(f)| = |X(-f)| and $\angle H(-f) = -\angle H(f)$ are not acceptable.

4. x(t) is a periodic signal with period T. The Fourier series expansion of x(t) is given by

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}, \quad f_0 = \frac{1}{T}$$

The autocorrelation of x(t) is defined as

$$R_x(\tau) = \frac{1}{T}\int_0^T x(t)x(t+\tau)dt$$

(a) Show that $R_x(\tau) = \sum_{n=-\infty}^{\infty} |X_n|^2 e^{j2\pi n f_0 \tau}$ (5%)

(b) The power spectral density of x(t), written symbolically as $S_x(f)$, is obtained as the Fourier transform of $R_x(\tau)$. Show that (5%)

$$S_x(f) = \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0)$$

(c) Show that $\int_{-\infty}^{\infty} S_x(f)df = \frac{1}{T}\int_0^T x^2(t)dt = P_x$, P_x is the average power of x(t).(5%)

5. (a) Find an LU-decomposition (or factorization) of (10%)

$$\begin{bmatrix} 2 & 4 & -1 & 5 \\ -4 & -5 & 3 & -8 \\ 2 & -5 & -4 & 1 \end{bmatrix}$$

(b) Does every square matrix have an LU-decomposition? Explain.(5%)

6.Let A and B be square matrices with the same size. Prove that if A is similar to B, then

(a) A^k is similar to B^k , where k is an integer.(5%)

(b) A and B have the same eigenvalues.(10%)