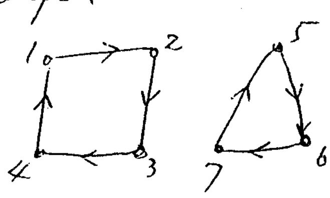
1. Let $f, g: Z^+ \rightarrow R$ where $f(n) = \sqrt{n}$ and $g(n) = (\log_2 n)^2$. Does $f \in O(g)$ or $g \in O(f)$? And prove it. (10%)

2. Let $R$ be the relation on $A = \{1, 2, 3, 4, 5, 6, 7\}$, where the directed graph associated with $R$ consists of the components shown in the following figure. Find the smallest integer $n > 1$, such that $R^n = R$. What is the smallest value of $n > 1$ for which the graph of $R^n$ contains some loops? Does it ever happen that the graph of $R^n$ consists of only loops? (10%)



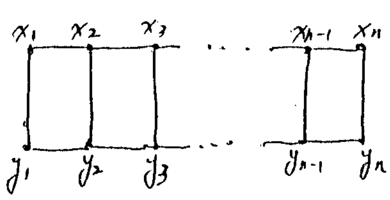

3. Let $m \in Z^+$ with $m$ odd. Prove that there exists a positive integer $n$ such that $m$ divides $2^n - 1$. (10%)

4. In how many ways can we select seven nonconsecutive integers from $\{1, 2, 3, \ldots, 50\}$. (10%)

5. For $n \geq 1$, $t_n = 1 + 2 + \cdots + n$. Find and solve a recurrence relation for $S_n$, $n \geq 1$, where $S_n = t_1 + t_2 + \cdots + t_n$. (10%)

6. (a) Let $k \in Z^+$, $k \geq 3$. If $G = (V, E)$ is a connected planar graph with $|V| = v$, $|E| = e$ and each cycle of length at least $k$, prove that $e \leq \left(\frac{k}{k-2}\right)(v-2)$.
(b) For a simple planar graph $G(V, E)$ with $|V| = 7$, $|E| = 15$, prove that each region of $G$ has exactly 3 edges. (10%)

7. Let $G(V, E)$ be the undirected connected ladder graph shown in the following figure. For $n \geq 0$, let $a_n$ be the number of spanning trees of $G$, whereas $b_n$ is the number of spanning trees that contain the edge $\{x_1, y_1\}$. a) Find a equation that expresses $a_n$ in terms of $a_{n-1}$ and $b_n$. b) Find another equation that expresses $b_n$ in terms of $a_{n-1}$ and $b_{n-1}$. c). Use the results in part (a) & (b) to set up and solve a recurrence relation for $a_n$. (10%)



8. Let $(R, +, \cdot)$ be the commutative ring with unity given by the following tables.

| + | z | u | a | b |
|---|---|---|---|---|
| z | z | u | a | b |
| u | u | z | b | a |
| a | a | b | z | u |
| b | b | a | u | z |

| $\cdot$ | z | u | a | b |
|---|---|---|---|---|
| z | z | z | z | z |
| u | z | u | a | b |
| a | z | a | b | u |
| b | z | b | u | a |

a) Is $R$ a field?
b) Find a subring of $R$ that is not an ideal.
c) Let $x$ and $y$ be unknowns. Solve the following system of equations in $R$:
$bx + y = u$; $x + by = z$. (10%)

9. In how many ways can we draw a diagonal on each face of a cube that is free to move in space. (10%)

10. If $G(V,E)$ is an undirected graph, a subset $I$ $(J)$ of $V$ $(E)$ is called an independent vertex (edge) set if no two vertices (edges) in $I$ $(J)$ are adjacent. An independent set $X$ is called maximal if no element $v$ can be added to $X$ with $X \cup \{v\}$ independent. The vertex (edge) independence number of $G$, denoted $\beta_0$ $(\beta_1)$, is the size of a largest vertex (edge) independent set in $G$. And, a subset $K$ $(L)$ of $V$ $(E)$ is called a vertex (edge) covering of $G$ if every edge $\{a,b\}$ (vertex $c$) of $G$ either $a$ or $b$ $(\exists y \in V, \{c,y\})$ is in $K$ $(L)$. The set $K$ $(L)$ is a minimal covering if $K$ $(L) - \{x\}$ fails to cover $G$ for each $x \in K$ $(L)$. The number of vertices (edges) in a smallest covering, denoted $\alpha_0$ $(\alpha_1)$, is called the vertex (edge) covering number of $G$. Please find the following graph's $\alpha_0$, $\beta_0$, $\alpha_1$, and $\beta_1$. (10%)