1. In how many ways can a positive integer n be written as a sum of r positive integer summands ($1 \le r \le n$) if the order of the summands is relevant? (10%)

2. If $\{x_1, x_2, \ldots, x_7\} \subseteq \mathbb{Z}^+$, prove or disprove that for some $i \neq j$, either $x_i + x_j$ or $x_i - x_j$ is divisible by 10. (10%)

3. If A and B are ideals of ring R, with glb$\{A, B\} = A \cap B$ and lub$\{A, B\} = A+B$, and define $A+B \triangleq \{a+b \mid a \in A, b \in B\}$. Prove that the poset formed by the ideals of R under set inclusion is a lattice. (10%)

4. Show that $d_n = n\, d_{n-1} + (-1)^n$ for $n \ge 1$, where d_n denotes the number of derangements of n objects. (10%)

5. How many r-digit quaternary sequences (sequences using only the digits 0, 1, 2, 3) have at least one 1, one 2, and one 3? (10%)

6. A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n and solve it. (15%)

7. Let $G = (V, E)$ be a loop free undirected graph. We call G color-critical if $\chi(G) > \chi(G-v)$ for all $v \in V$, where $\chi(G)$ denotes the chromatic number of G. (10%)
   a). For $n \in \mathbb{Z}^+$, $n \ge 2$, which of the complete graphs K_n are color-critical?
   b). Prove that a color-critical graph must be connected.

(背面仍有題目,請繼續作答)

8. Let $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Q}$, where $\mathbb{Z}$ is the set of integers and $\mathbb{Q}$ is the set of rational numbers. Define the addition ($\oplus$) and multiplication ($\odot$) for R as follows: For $(a, b, q), (c, d, r) \in R$, $(a, b, q) \oplus (c, d, r) \triangleq (a+c, b+d, q+r)$ and $(a, b, q) \odot (c, d, r) \triangleq (ac, bd, qr)$. (10%)

a). Characterize when an element (m, n, s) of R is a proper divisor of zero, and give an example.

b). Characterize when an element (m, n, s) of R is a unit.

9. Construct a state diagram for a finite state machine that recognizes the input string $x = 1011$. It is also required to recognize overlapping sequences, as can be seen in the output z that results from the following input string x: (15%)

$x = 1011011001011 00$

$z = 0001001000001 00$