1. Solve for the given differential equations.
   (a) $(1-x^2)y''-2xy'+2y=0, \quad -1<x<1.$ (7%)
   (b) $y''-3y'-4y=\dfrac{e^{4x}(5x-2)}{x^3}.$ (7%)
   (c) $yy''+(y+1)(y')^2=0.$ (6%)

2. Use the Laplace transform to solve Bessel's equation of order zero. (10%)
   $ty''+y'+ty=0; \; y(0)=1.$

3. Solve for the integral equation. (5%)
   $f(t)=\cos(t)+e^{-2t}\int_0^t f(a)e^{2a}da$

4. (a) Find a Fourier series of period 6 which in the interval (1, 7) represents a function $f(x)$ taking on the constant value +1 when $1<x<4$ and constant value −1 when $4<x<7$.
   (b)Reducing the above Fourier series to the following form:
   $$f(x)=A\sum_{n\text{ odd}} B\sin\frac{n\pi(x-1)}{3}$$
   What are the values of A and B? (12%)

5. Suppose that the analytic function $f(z)$ has a pole of order m at the point $z=a$, derive the formula for evaluating the residue $\mathrm{Res}[f(z);a]$. (8%)

6. Evaluate the following integral (15%)
   $$\int_{-1}^{1}\frac{z+1}{z^2}dz$$
   (a) If the path is the upper half of the circle $r=1$,
   (b) If the path is the lower half of the circle $r=1$,
   (c) Explain the solutions you have obtained in (a) and (b).

7. Consider a first-order matrix equation $\dot{X}(t)=AX(t)$, where $A=\begin{bmatrix}3 & 4\\ 3 & 2\end{bmatrix}$. (a) Find the eigenvalues and corresponding eigenvectors of A. (b) Solve $X(t)$ if $X(0)=\begin{bmatrix}6\\ 1\end{bmatrix}$. (10%)

8. Let $\boldsymbol{B}$ be an $m\times n$ matrix. If $\boldsymbol{A}$ is a nonsingular $m\times m$ matrix, show that $\boldsymbol{AB}$ and $\boldsymbol{B}$ have the same nullspace and hence the same rank. (10%)

9. Let $\boldsymbol{B}$ be a symmetric tridiagonal matrix (*i.e.*, $\boldsymbol{B}$ is symmetric and $b_{ij}=0$ whenever $|i-j|>1$). Let $\boldsymbol{C}$ be the matrix formed from $\boldsymbol{B}$ by deleting the first two rows and columns. Show that $\det(\boldsymbol{B})=b_{11}B_{11}-b_{12}^2\det(\boldsymbol{C})$, where B_{11} is the cofactor of b_{11}. (10%)