系所組別： 製造資訊與系統研究所甲組
考試科目： 工程數學
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Problem 1 (15 points)

A model for the populations of two interacting species of animals is

$$\begin{aligned}\frac{dx}{dt} &= k_1 x(\alpha - x),\\ \frac{dy}{dt} &= k_2 xy,\end{aligned}$$

where $k_{1,2}$ and α are positive constants.

(a) Find the equilibrium points of (x, y), and determine their stability.

(b) For the initial conditions $x(0) = x_0$ and $y(0) = y_0$, determine $x(t)$ and $y(t)$.

Problem 2 (10 points)

Consider the boundary-value problem (BVP) introduced in the construction of the mathematical model for the shape of a rotating string:

$$T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0; \quad y(0) = 0, \quad y(L) = 0.$$

For constant T and ρ, the critical angular speeds ω_n are the values of ω for which the BVP has nontrivial solutions. Find the critical angular speeds ω_n and the corresponding deflections $y_n(x)$.

Problem 3 (15 points)

Use *Laplace transform* to solve the following initial-value problem:

$$\ddot{y} + 16y = f(t), \quad y(0) = 0, \quad \dot{y}(0) = 1,$$

where

$$f(t) = \begin{cases} \cos 4t, & 0 \le t < \pi, \\ 0, & t \ge \pi. \end{cases}$$

Problem 4 (15 points)

Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} 5 & -1 & 0 \\ 0 & -5 & 9 \\ 5 & -1 & 0 \end{pmatrix}.$$

(背面仍有題目,請繼續作答)

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Problem 5 (10 points)

Suppose there is a continuous distribution of charge throughout a closed and bounded region D enclosed by a surface S. Then the natural extension of Gauss' law is given by

$$\iint_S (\mathbf{E}\cdot\mathbf{n})\,dS = \iiint_D 4\pi\rho\,dV,$$

where $\rho(x,y,z)$ is the charge density or charge per unit volume, and $\mathbf{E}(x,y,z)$ is the electric field.

(a) Show that $\nabla\cdot\mathbf{E} = 4\pi\rho$.

(b) Suppose that $\mathbf{E}$ is an irrotational vector field. What equation does the potential ϕ for $\mathbf{E}$ satisfy?

Problem 6 (25 points)

The transverse displacement $u(x,t)$ of a vibrating beam of length L is determined from a fourth-order partial differential equation

$$a^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0 \quad (0 < x < L,\ t > 0).$$

If the beam is simply supported, the boundary and initial conditions are

$$\begin{aligned} &u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,\\ &\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0, \quad t > 0;\\ &u(x,0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L. \end{aligned}$$

Solve for $u(x,t)$.

Problem 7 (10 points)

Use Cauchy's residue theorem to evaluate the following integral along the indicated contour:

$$\oint_C \frac{2z+5}{z(z+2)(z-1)^4}\,dz, \quad C: |z+2| = \frac{5}{2} \text{ (counterclockwise)}.$$