Linear Algebra (50%)

Let $A = [a_{i,j}]$ be an $n \times n$ square matrix. Let $\vec{x} = [x_1, x_2, \cdots, x_n]^t$ be an $n \times 1$ column vector of unknown mathematical variables and $\vec{b} = [b_1, b_2, \cdots, b_n]^t$ is an $n \times 1$ column vector of constants, where the supscript t indicates transpose. Then, $A\vec{x} = \vec{b}$ represents a set of linear equations. Let rank(A) denote the rank of matrix A. Answer the following questions:

1. (10%) Give the necessary and sufficient condition(s) that the linear equations have exactly one solution.
2. (10%) Give the necessary and sufficient condition(s) that the linear equations have more than one solution.
3. (15%) Is it possible that the linear equations have exactly two distinct solutions. If the answer is yes, give a numerical example of A.
4. (15%) Is it possible that the linear equations have no solution at all. If the answer is yes, give a numerical example of A.

Probability and Statistics (50%)

Answer the following questions:

1. (10%) Let $X_1, X_2, \cdots, X_n$ be n independently and identically distributed random variables of mean μ and variance σ^2. Let $\overline{X} = \sum_{i=1}^{n} X_i/n$. Give the mean of $\overline{X}$ and the variance of $\overline{X}$.
2. (10%) Let X be a random variable having a mean μ and a variance σ^2. Let k be some positive number. Then, Chebyshev's inequality states that

$$P(|X - \mu| \geq k \cdot \sigma) \leq \frac{1}{k^2}$$

   Give all the condition(s) under which you may want to apply the above Chebyshev's inequality.

3. (10%) What is the statistical simulation for?
4. (20%) When do we have no other choice but conducting the simulation? You may use an illustration example to explain your answer.