Solve the following differential equations:

1. (a) Find the general solution of $x^3 \frac{dy}{dx} = x^2 y - y^3$

   (5%)

   (b) Find the particular solution of $3y^4 - 1 + 12xy^3 \frac{dy}{dx} = 0; y(1) = 2$

   (5%)

2. (a) Find the particular solution of $y'' + 8y' + 16y = 0; y(0) = y'(0) = 3$

   (7%)

   (b) Find the general solution of $yy'' + (y')^2 = 0$

   (8%)

3. Find the general solution of $y'' - 3y' = 2e^{2x}\sin(x); y(0) = 1, y'(0) = 2$

   (10%)

4. Solve the following boundary value problem:
   (15%)

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}; (0 < x < 2, t > 0)$$

with

$$y(0,t) = y(2,t) = 0, (t > 0)$$
$$y(x,0) = 0$$
$$\frac{\partial y}{\partial t}(x,0) = 2x, (2 > x > 0)$$

(背面仍有題目,請繼續作答)

5. Show that the quantities 1, x, and x^2 are linearly independent. (10%)

6. Show that the following two quadratic forms are positive-definite, negative-definite, positive-semidefinite, negative-semidefinite, or indefinite. (10%)

i) $x_1^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_1 + 5x_2^2 - 4x_2x_3 - 2x_3x_1 - 4x_3x_2 + 5x_3^2$.

ii) $x_1^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_1 + 3x_2^2 - 4x_2x_3 - 2x_3x_1 - 4x_3x_2 + 5x_3^2$.

7. Let (15%)

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find e^{At} by using the formula $\mathcal{L}[e^{At}] = (sI - A)^{-1}$.

8. Consider the dynamical equation (15%)

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

Is it possible to choose an initial state at $t = 0$ such that the output of the dynamical equation is of the form $y(t) = te^{-t}$ for $t > 0$? (You must show your work.)