

Problem A1. (25%)

Give an  $n \times n$  square matrix  $A$ . You are asked to identify an  $n \times n$  square matrix  $E$  such that  $E'AE = I$ ,  $E'$  is the transpose of  $E$ , and  $I$  is the  $n \times n$  identity matrix. Answer the following questions:

1. (5%) Give the necessary and sufficient condition that no such  $E$  matrix exists.
2. (15%) Give a procedure which generates  $E$  when such an  $E$  matrix exists.
3. (5%) Is such an  $E$  matrix unique when one does exist? Just answer "yes" or "no". You do not have to offer any explanation.

Problem A2. (25%)

Answer the following questions:

1. (10%) Let  $X$  be a random variable having a mean  $\mu$  and a variance  $\sigma^2$ . Let  $k$  be some positive real number. The Chebyshev's inequality states that

$$P(|X - \mu| \geq k \cdot \sigma) \leq \frac{1}{k^2}. \quad (1)$$

While the above inequality (1) holds true, is the following inequality (2) also true? Briefly explain your answer.

$$P(|X - \mu| \leq k \cdot \sigma) \geq 1 - \frac{1}{k^2}. \quad (2)$$

2. (15%) Let  $f(x)$  be the probability density function of random variable  $X$ , where  $f(x) = 1/3$  for  $-1 \leq x \leq 2$  and  $f(x) = 0$  while  $x < -1$  or  $x > 2$ . Let  $g(y)$  be the probability density function of random variable  $Y$ , where  $Y = X^2$ . Derive  $g(y)$ .

(背面仍有題目,請繼續作答)

B - (1) Let  $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$  Find  $AA^t$ . (5%)

B - (2) Find the rank of matrix  $A = \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$ . (6%)

B - (3) Find eigenvalues and associated nonzero eigenvectors of the matrix  $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ . (7%)

B - (4) Let the probability distribution of a random variable  $X$  is as follow:

|           |     |     |   |     |
|-----------|-----|-----|---|-----|
| $x$       | 0   | 1   | 2 | 3   |
| $Pr(X=x)$ | 1/3 | 1/2 | 0 | 1/6 |

Compute  $E\{(X-E(X))^2\}$ . (8%)

B - (5) The probability density of a continuous random variable  $X$  in between  $x=2$  and  $x=5$  is  $f(x) = 2(1+x)/27$ . Compute  $Pr(3 \leq X < 4)$ . (8%)

B - (6)  $\lim_{t \rightarrow 0} \frac{\cos t - 1}{t} = ?$  (8%)

B - (7) Evaluate  $\int_0^2 \int_0^{\sqrt{4-x^2}} \exp(-x^2) \exp(-y^2) dy dx$ . (8%)