

Problem 1 (15 points)

Solve the following initial value problem:

$$y''' - 4y' = 8e^{2x} + 5 \sin x; \quad y(0) = 1, \quad y'(0) = 3, \quad y''(0) = -1.$$

Problem 2 (10 points)

Suppose that the differential equation $x^2 y'' - 3xy' + by = 0$ has as a particular solution $y_p = x^2$. Find the general solution of the above differential equation.

Problem 3 (15 points)

(a) Solve the linear system of equations

$$\begin{aligned} 5x + 5y - 10z &= 0, \\ 2w - 3x - 3y + 6z &= 2, \\ 4w + x + y - 2z &= 4. \end{aligned}$$

(b) Calculate rank \mathbf{A} and nullity \mathbf{A} , where \mathbf{A} is the coefficient matrix of the above linear system.

Problem 4 (10 points)

Show that for any non-singular $n \times n$ matrix \mathbf{A} , (a) $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$; and (b) $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$.

Problem 5 (10 points)

As a physical fact, the flow of heat in a temperature field takes place in the direction of maximum temperature decrease. Suppose now that a temperature field is described by $T(x, y) = \cos x \cosh y$. Find the direction of heat flow at the point $P: (\pi/2, 1)$.

Problem 6 (15 points)

Use Laplace transform to solve the integral equation

$$y(t) = te^t - 2e^t \int_0^t e^{-\tau} y(\tau) d\tau.$$

Problem 7 (15 points)

Find the Fourier series of the periodic function

$$f(x) = |\sin x|, \quad (-\pi < x < \pi); \quad f(x) = f(x + 2\pi).$$

Problem 8 (10 points)

Evaluate $\oint_C \frac{\sinh 2z}{z^4} dz$, taking the contour C to be the unit circle $|z| = 1$ (counterclockwise).