※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. Solve the following ordinary differential equations and find the solutions:

   (6%) (a) $y''+2y'-3y=2e^{x},\quad y(0)=4,\quad y'(0)=2.$

   (6%) (b) $y''-5y'+6y=2\sin^{2}(4x).$

   (6%) (c) $x^{2}y''-5xy'+10y=0.$

2. (12%) Prove that: if **A** is a real symmetric matrix, then its eigenvalues are real, and the eigenvectors associated with distinct eigenvalues are orthogonal.

3. (20%) A uniform beam of length L carries a concentrated load w at $x = L/2$. The beam is embedded at its left end and is free at its right end. Use the **Laplace transform** to determine the deflection $y(x)$ from the beam equation:

$$EI\frac{d^4y}{dx^4}=w\delta(x-L/2),$$

with the boundary conditions $y(0)=0$, $y'(0)=0$, $y''(L)=0$, and $y'''(L)=0$. In the above equation, EI is the flexural rigidity of the beam, and $\delta(\cdot)$ is the Dirac delta function.

Hint: First solve the beam equation for $x\in[0,\infty)$ with $y(0)=0$, $y'(0)=0$, and unspecified $y''(0)$ and $y'''(0)$. Then choose $y''(0)$ and $y'''(0)$ such that $y''(L)=0$ and $y'''(L)=0$.

4. (15%) The Joukowski transformation

$$z=\zeta+c^2/\zeta,$$

with c being a real constant, is the best known in aerodynamics.

(a) Show that the exterior of the circle $|\zeta|=c$ in the ζ plane is mapped onto the entire z plane cut along the real axis $-2c\le x\le 2c,\quad y=0$.

(b) Also show that the point ζ outside the circle $|\zeta|=c$ and the image point c^2/ζ inside are mapped onto the same point in the z plane.

(c) Now consider a circle $|\zeta|=c'$ with $c'>c$. Determine its mapped image in the z plane.

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5. (15%) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(mx)dx}{1+x^2}$

6. (20%) Determine the solution T(x, t) of the following partial differential equation.

$$\frac{\partial^2 T}{\partial t^2} = c^2 \frac{\partial^2 T}{\partial x^2} \quad \text{in} \quad 0 < x < \pi \quad \text{and} \quad t > 0$$

with the boundary conditions

$$T(0, t) = 0 \quad \text{and} \quad T(\pi, t) = 0$$

and the initial conditions

$$T(x,0) = 1, \qquad \frac{\partial T}{\partial t}(x,0) = 0$$