

1. Solve the initial value problem (10%)

$$\frac{dy}{dx} = \sqrt{2gy}, \quad \text{with } y(x) = 0 \text{ at } x = 0.$$

where g is constant.

2. Let $\vec{F} = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$ a) Calculate $\nabla \times \vec{F}$.

(b) Evaluate $\oint \vec{F} \cdot d\vec{r}$ around any closed path. (13%)

3. Evaluate $\int_c z^2 \exp(2/z) dz$

where c is a circle of radius 2, centered at the origin. (10%)

4. Solve the Laplace's Equation (10%)

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = 0 \quad \text{in the circle } 0 \leq r \leq a$$

Subject to the Boundary Condition

$$T(a, \theta) = f(\theta) \quad \text{on } r = a \quad \text{where } f(\theta) \text{ is given.}$$

5. (a) Find the Laplace transform of $\mathcal{L}[S_\epsilon(t)] = ?$

$$S_\epsilon(t) = \begin{cases} 0 & \text{for } -\alpha < t < 0 \\ \epsilon^{-1} & \text{for } 0 < t < \epsilon \\ 0 & \text{for } \epsilon < t < \alpha \end{cases}$$

and if $\epsilon \rightarrow 0$

(b) $\mathcal{L}[e^{-2t} t^{-1} \sin t] = ?$ (12%)

6. 已知矩陣 $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (10%)

試求對應 A 矩陣之特徵值及特徵向量。

7. Consider the eigenvalue problem (12%)

$$x^2 \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} + \lambda \phi = 0$$

with Boundary Conditions: $\phi(1) = 0$, $\phi(6) = 0$

a) Show that multiplying by $1/x$, puts this in Sturm-Liouville form.

b) Show that $\lambda \geq 0$

c) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

8. (a) 試求函數 $f(x) = e^{-2x}$ 之傅立葉積分 (Fourier integral)
此外 $f(x)$ 滿足下列之性質:

當 $x > 0$ 時 $f(-x) = -f(x)$ (8%)

(b) 試求 $\int_0^{\infty} \frac{\omega \sin 3\omega \cos \omega}{4 + \omega^2} d\omega$ 之值. (5%)

9. 試求下列偏微分方程式之解: (10%)

$$\frac{\partial^2 y}{\partial x^2} + u(x-0.5) = \frac{\partial y}{\partial t}, \quad 0 \leq x \leq 1, \quad t > 0$$

起始條件: $y(x, 0) = 0$

邊界條件: $y(0, t) = y(1, t) = 0$

式中 $u(x-0.5)$ 為 unit step function.