1. (a) Consider a system of differential equations $\underline{X}' = \underline{A}\,\underline{X}$, where $\underline{A}$ is an $n \times n$ constant matrix with real elements. Suppose that $\lambda = \alpha + i\beta$ is an eigenvalue of $\underline{A}$, with a corresponding eigenvector $\underline{\xi} = \underline{U} + i\,\underline{V}$. Show that

$$e^{\alpha t}\left(\underline{U}\cos(\beta t) - \underline{V}\sin(\beta t) \right)$$

and

$$e^{\alpha t}\left(\underline{U}\sin(\beta t) + \underline{V}\cos(\beta t) \right)$$

are linearly independent solutions of $\underline{X}' = \underline{A}\,\underline{X}$.

(b) Find the real-valued general solution of the system $\underline{X}' = \underline{A}\,\underline{X}$ with

$$\underline{A} = \begin{bmatrix} 6 & -5 \\ 5 & -2 \end{bmatrix}$$

by using the result of (a). (15%)

2. Solve the problem:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + Ax \qquad (0 < x < L,\ t > 0),$$

$$y(0,t) = y(L,t) = 0 \qquad (t > 0),$$

$$y(x,0) = 0 \qquad (0 < x < L),$$

$$\frac{\partial y}{\partial t}(x,0) = 0 \qquad (0 < x < L).$$

(20%)

3. 已知函數 $f(x,y,z) = xy + yz + zx$，試求出下列各種情況之結果. (15%)

(a) 求 df/ds 在点 $(1,1,3)$ 之值，其中 S 為朝向点 $(1,1,1)$ 之路径.

(b) 求 df/ds 在点 $(1,1,3)$ 之最大值，並求出在此情況下之方向.

(c) 求垂直於平面 $xy + yz + zx = 7$ 之向量.

4. 試計算对应於下列各種曲線 C 之積分 $\oint_C \frac{z^2-1}{z^2+1}\,dz$ 之值，其中曲線 C 分別為:

(a) $|z-1| = 1$. (b) $|z-i| = 1$. (c) $|z+i| = 1$

(d) $|z-2i| = 2$. (e) $|z| = 2$ (20%)

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5 (a) Find the inverse of the given Laplace Transform

$$F(s) = \ln\left(1 + \frac{4}{s^2}\right) \qquad (5\%)$$

(b) Using the Convolution theorem to Compute the inverse Laplace Transform of

$$F(s) = \frac{2}{s^3(s^2+1)} \qquad (5\%)$$

(c) solve the differential equation

$$y''(t) + 2y'(t) + y(t) = \delta(t-1)$$

when $y(0) = 2$, $y'(0) = 3$ (10%)

6 Show that Chebyshev's equation

$$(1-x^2)y'' - xy' + n^2 y = 0 \qquad \text{where } n = 1, 2, 3 \cdots$$

is of Sturm-Liouville type and determine the weight function for which the resulting polynomials $y_0 = 1$, $y_1 = x$, $y_2 = 2x^2 - 1$

$y_3 = 4x^3 - 3x$, $y_4 = 8x^4 - 8x^2 + 1$, ...

are orthogonal. (10%)