1. The functions

$$y_1 = x^3 \quad \text{and} \quad y_2 = \begin{cases} x^3 & \text{when } x \ge 0 \\ -x^3 & \text{when } x \le 0 \end{cases}$$

are two linearly independent solutions of the differential equation

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 0 \qquad \text{for all } x.$$

(a) Find the Wronskian of y_1 and y_2. (5%)

(b) Does the result of (a) contradict the linear independence of y_1 and y_2? Why? (10%)

2. We have the equation

$$\mathcal{F}\left[\frac{d^2 f}{dx^2}\right] = -\omega^2 \mathcal{F}[f]$$

for the Fourier transform of the second derivative of $f(x)$. The condition $f(x) \to 0$ for $x \to \pm\infty$ may be relaxed slightly. Find the least restrictive condition for the above equation to hold. (10%)

3. **<u>Don't need to show your solution procedures, just give your answer only.</u>**

(a) What is the directional derivative of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at the point P: (2, 1, 3) in the direction of the vector $\vec{a} = \vec{i} - 2\vec{k}$?(5 %)

(b) What is the unit normal vector of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point P: (1, 0, 2) ?(5 %)

(c) Evaluate $\iint_S (7x\vec{i} - z\vec{k}) \bullet \vec{n}\, dA$ over S: $x^2 + y^2 + z^2 = 4$ by the divergence theorem.(5 %)

4. **<u>Don't need to show your solution procedures, just give your answer only.</u>**

(a) Any real square matrix A can be written as the sum of a symmetric R and a skew-symmetric matrix S. Now A is given as below, what are R and S ?

$$A = \begin{bmatrix} 3 & -4 & -1 \\ 6 & 0 & -1 \\ -3 & 13 & -4 \end{bmatrix} \qquad (5\ \%)$$

(b) What are the eigenvalues and corresponding eigenvectors of a matrix $A = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$?(5 %)

(背面仍有題目,請繼續作答)

5. Suppose $f(t)$ satisfies the difference-differential equation

$$\frac{df(t)}{dt}+f(t)-f(t-1)=0, \qquad t\geq 0,$$

and the initial condition, $f(t)=f_0(t)$, $-1\leq t\leq 0$, where $f_0(t)$ is given. Show that the Laplace transform $F(s)$ of $f(t)$ satisfies

$$F(s)=\frac{f_0(0)}{1+s-e^{-s}}+\frac{e^{-s}}{1+s-e^{-s}}\int_{-1}^{0}e^{-st}f_0(t)\,dt.$$

Find $f(t)$, $t\geq 0$, when $f_0(t)=1$. Check the result. (13%)

6. Apply Cauchy's theorem to $f(z)=e^{-z^2}$ using the contour C consisting of the boundary of the sector formed from the positive real axis, the arc $|z|=R$, $0\leq\theta\leq\pi/4$, and the line through the origin making an angle $\pi/4$ with the real axis. Let $R\to\infty$ to show that

$$\int_0^\infty \sin x^2\,dx=\int_0^\infty \cos x^2\,dx=\sqrt{\pi/8}.$$

[You may assume $\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}$.] (12%)

7. Please find solution of the following partial differential equation:

$\nabla^2 u=0$	$(0<x<a,\ 0<y<b)$	
$u(x,0)=u(x,b)=0$	$(0<x<a)$	
$u(0,y)=0$	$(0<y<b)$	
$u(a,y)=T$	$(0<y<b)$	(15%)

8. Consider the regular Sturm-Liouville problem on [0, 1]:

$$y''+\lambda y=0;\ \ y(0)=0,\ \ 2y(1)+y'(1)=0$$

Discuss cases on λ. (10%)